ISRAEL JOURNAL OF MATHEMATICS 99 (1997), 231–270

GRADED METRICS ADAPTED TO SPLITTINGS*

BY

J. MONTERDE

Departament de Geometria i Topologia Facultat de Matemàtiques
Universitat de València
C/Dr. Moliner 50, 46100–Burjassot (València), Spain
e-mail: monterde@iluso.ci.uv.es

AND

O.A. SÁNCHEZ-VALENZUELA

Centro de Investigación en Matemáticas
Apdo. Postal, 402; C.P. 36000 Guanajuato, Gto., México
e-mail: saval@servidor.dgsca.unam.mx

ABSTRACT

Homogeneous graded metrics over split $\mathbb{Z}_2$-graded manifolds whose Levi-Civita connection is adapted to a given splitting, in the sense recently introduced by Koszul, are completely described. A subclass of such is singled out by the vanishing of certain components of the graded curvature tensor, a condition that plays a role similar to the closedness of a graded symplectic form in graded symplectic geometry: It amounts to determining a graded metric by the data $\{g, \omega, \nabla'\}$, where g is a metric tensor on M, ω is a fibered nondegenerate skewsymmetric bilinear form on the Batchelor bundle $E \to M$, and ∇' is a connection on E satisfying $\nabla'\omega = 0$. Odd metrics are also studied under the same criterion and they are specified by the data $\{\kappa, \nabla'\}$, with $\kappa \in \mathrm{Hom}(TM, E)$ invertible, and $\nabla'\kappa = 0$. It is shown in general that even graded metrics of constant graded curvature can be supported only over a Riemannian manifold of constant curvature, and the curvature of ∇' on E satisfies $R^{\nabla'}(X,Y)^2 = 0$. It is shown that graded Ricci flat even metrics are supported over Ricci flat manifolds and the curvature of the connection ∇' satisfies a specific set of equations. Finally, graded Einstein even metrics can be supported only over Ricci flat

* Partially supported by DGICYT grants #PB94-0972, and SAB94-0311; IVEI grant 95 – 031; CONACyT grant #3189-E9307.
Received August 27, 1995

Riemannian manifolds. Related results for graded metrics on $\Omega(M)$ are also discussed.

Introduction and results

BACKGROUND. Let M be an m-dimensional smooth manifold, and let C^∞_M be the sheaf of smooth functions on M. Let $E \to M$ be a vector bundle of rank n, and let $\mathcal{E}$ be its sheaf of smooth sections. Let $\wedge\mathcal{E}$ be the sheaf of smooth sections of the exterior algebra bundle $\wedge E \to M$. This is a sheaf of $\mathbb{Z}_2$-graded (graded-commutative) algebras over M, and the ringed space $(M, \wedge\mathcal{E})$ is a **split graded manifold** of dimension (m, n). Abstractly, a smooth graded manifold is defined as a ringed space $(M, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded (graded-commutative) algebras over a smooth manifold M, and one is given the exact sequence

$$0 \to \mathcal{N} \to \mathcal{A} \to \mathcal{A}/\mathcal{N} \to 0$$

in which $\mathcal{N}$ is the nilpotent ideal of $\mathcal{A}$, and $\mathcal{A}/\mathcal{N}$ is the sheaf C^∞_M (regarded as trivially graded). Holomorphic graded manifolds are defined similarly. Smooth graded manifolds are split; i.e., there exists a smooth vector bundle $E \to M$ such that $\mathcal{A} \simeq \wedge\mathcal{E}$ (see [1]). By way of contrast, holomorphic graded manifolds are not always split (see [3], and [10]). Also, there are obstructions to split smooth graded manifolds equivariantly under the action of a given Lie group [12].

A new and enlightening characterzation of split graded manifolds has recently been given by Koszul in [7]. He has shown that a graded manifold splits if and only if ($\mathbb{Z}_2$-graded) connections exist, and that such exist, if and only if some special derivations of the structure sheaf $\mathcal{A}$—called **adapted derivations**—exist. Adapted derivations potentially produce a $\mathbb{Z}$-grading out of the given $\mathbb{Z}_2$-graded structure in the following manner: The filtration of $\mathcal{A}$ defined by the powers $\mathcal{N}^r \supset \mathcal{N}^{r+1}$ $(r \geq 0)$ yields a filtration $(\operatorname{Der}\mathcal{A})^r \supset (\operatorname{Der}\mathcal{A})^{r+1}$ $(r \geq -1)$ in the sheaf of $\mathbb{Z}_2$-graded derivations $\operatorname{Der}\mathcal{A}$ by letting a section D be in $(\operatorname{Der}\mathcal{A})^r$ if and only if $D\mathcal{N}^p \subset \mathcal{N}^{p+r}$ for all $p \geq 0$. An even derivation H is called adapted to this filtration if $(H - r\mathrm{Id})\mathcal{N}^r \subset \mathcal{N}^{r+1}$; this also implies that $(\mathrm{ad}(H) - r\mathrm{Id})(\operatorname{Der}\mathcal{A})^r \subset (\operatorname{Der}\mathcal{A})^{r+1}$. For finite-dimensional graded manifolds, there exists a natural number n—the odd dimension—such that $\mathcal{N}^{n+1} = 0$.

When an adapted derivation exists, it may be used as in [7] to produce a non-trivial morphism (**splitting**) of $\mathbb{Z}_2$-graded $\mathcal{A}$-modules,

$$\mathcal{A} \leftarrow \mathcal{A}/\mathcal{N} + \mathcal{N}/\mathcal{N}^2,$$

which fits into the diagram

$$\mathcal{A} \to \mathrm{Gr}\mathcal{A} = \sum_{r\geq 0} \mathcal{N}^r/\mathcal{N}^{r+1} \hookleftarrow \mathcal{A}/\mathcal{N} + \mathcal{N}/\mathcal{N}^2$$

thus inducing on $\mathcal{A}$ the $\mathbb{Z}$-grading of $\mathrm{Gr}\mathcal{A}$ via the universal extension of the given splitting. Now, $\mathcal{N}/\mathcal{N}^2$ has the structure of a locally free sheaf of $\mathcal{A}/\mathcal{N}$-modules; that is, a vector bundle (the so-called Batchelor bundle) whose rank is equal to the odd dimension. We shall write $\mathcal{E}$ instead of $\mathcal{N}/\mathcal{N}^2$ when we think of it as the sheaf of sections of the corresponding vector bundle $E \to M$. Since $\mathrm{Gr}\mathcal{A} = \wedge\mathcal{E}$, a splitting yields a morphism, $(M, \mathcal{A}) \to (M, \wedge\mathcal{E})$ which is inverse to the natural map $(M, \wedge\mathcal{E}) \to (M, \mathcal{A})$ defined by gr: $\mathcal{A} \to \mathrm{Gr}\mathcal{A}$.

Metrics adapted to a splitting. The main point of [7] on which we have based our work is the relationship between adapted derivations and connections: Given a graded connection ∇: $\mathrm{Der}\,\mathcal{A} \times \mathrm{Der}\,\mathcal{A} \to \mathrm{Der}\,\mathcal{A}$, there exists a unique adapted derivation H^{∇}, having the property that, $\nabla_{H^{\nabla}} H^{\nabla} = H^{\nabla}$. Conversely, given an adapted derivation H, there exists a graded connection ∇ whose H^{∇} is equal to H. As we shall only deal with split graded manifolds, we assume that the sheaf of $\mathbb{Z}_2$-graded algebras over M that we are given is $\wedge\mathcal{E}$. The splitting (or $\mathbb{Z}$-grading) of $\wedge\mathcal{E} = \bigoplus_{k\geq 0} \wedge^k\mathcal{E}$ is produced by the adapted derivation H defined as the unique degree-preserving derivation $\wedge\mathcal{E} \to \wedge\mathcal{E}$ which is zero on $\wedge^0\mathcal{E}$, and the identity on $\wedge^1\mathcal{E}$. Such an H is then uniquely characterized by the property that $H|_{\wedge^k\mathcal{E}} = k\mathrm{Id}|_{\wedge^k\mathcal{E}}$, for each k. We call it **the canonical splitting** of $\wedge\mathcal{E}$. Also, it shall be assumed throughout this paper that $\mathrm{Der}\,\wedge\mathcal{E}$ has the $\mathbb{Z}$ grading defined by H: The $\mathbb{Z}$-degree $|D|$ of a $\mathbb{Z}$-homogeneous element D makes sense only for derivations D such that $\mathrm{ad}(H)D = |D|D$.

Our purpose is to characterize the class of homogeneous $\mathbb{Z}_2$-graded metrics $\langle\, \cdot\,, \cdot\,\rangle$: $\mathrm{Der}\,\wedge\mathcal{E} \times \mathrm{Der}\,\wedge\mathcal{E} \to \wedge\mathcal{E}$ whose Levi-Civita graded connection ∇ satisfies $\nabla_H H = H$ for the canonical splitting H of $\wedge\mathcal{E}$. We shall say that these metrics are **adapted to the canonical splitting** of $\wedge\mathcal{E}$. When a connection on the bundle $E \to M$ is given, such graded metrics are described by a set of sections from $S^2(\mathrm{Der}\, C^\infty_M)^* \otimes \wedge\mathcal{E}$, $(\mathrm{Der}\, C^\infty_M)^* \otimes \mathcal{E} \otimes \wedge\mathcal{E}$, and $(\wedge^2\mathcal{E}) \otimes \wedge\mathcal{E}$ having some definite symmetries and satisfying some specific relations between them (Proposition 2.3 below). The automorphism group of the $\mathbb{Z}_2$-graded algebra $\wedge\mathcal{E}$ acts on the space of graded metrics and it is shown (Theorem 2.4) that the description of Proposition 2.3 is complete up to the action of $\mathrm{Aut} \wedge \mathcal{E}$.

In order to get a better hold on the structure of these metrics we restrict our attention to the subclass of those having **second order depth**; this means that a graded basis $\{D_\alpha\}$ of $\mathrm{Der} \wedge\mathcal{E}$ exists for which

$$\langle D_\alpha, D_\beta\rangle \in \bigoplus_{k\leq 2} \wedge^k\mathcal{E}.$$

It is shown (Proposition 3.1) that this subclass is characterized by the fact that the endomorphism $\mathrm{Der} \wedge\mathcal{E} \ni D \mapsto R(H, D)H \in \mathrm{Der} \wedge\mathcal{E}$—$R$ being the graded curvature of the Levi-Civita graded connection—vanishes identically. Furthermore, it is shown (Proposition 4.1) that for such metrics a connection ∇' on E can be chosen to completely describe them by fewer tensors: If the graded metric is even, it requires an ordinary metric tensor g on M and a non-degenerate, skewsymmetric, bilinear (fibered) form $\omega \in \wedge^2\mathcal{E}$, with $\nabla'\omega = 0$. In short, an even adapted metric of this sort is characterized by the data $\{g, \omega, \nabla'\}$. Moreover, ω is actually determined by H, since (Proposition 2.1)

$$\omega = c\langle H, H\rangle \qquad (c \text{ a constant factor}).$$

When odd adapted metrics exist ($\dim E = \dim M = 2n$) the data is $\{\kappa, \nabla'\}$, where $\kappa \in (\mathrm{Der}\, C^\infty_M)^* \otimes \mathcal{E}$ defines a non-degenerate bilinear pairing and ∇' is a connection on E chosen in such a way that the subspace $(\mathrm{Der} \wedge\mathcal{E})_0$ of degree-zero derivations becomes totally isotropic. These results are to be contrasted with those corresponding to graded symplectic structures on split graded manifolds $(M, \wedge\mathcal{E})$ (see [8] and [11]): The fact that a graded symplectic form is closed makes it possible to always characterize it by reduced data of this sort. In the graded Riemannian case the condition $R(H, \cdot\,)H = 0$ is what allows the reduction.

To explain how close this relationship really is we recall the main result from [11]: Let $E \to M$ be a vector bundle over a symplectic manifold M with symplectic form ω. Let ∇ be a connection on $\wedge E$ and let g be a fibered metric on $\wedge E$ compatible with ∇. The data $\{g, \omega, \nabla\}$ uniquely defines a graded symplectic structure on $(M, \wedge\mathcal{E})$. Furthermore, its structure is of **second order depth**: there exists a graded basis $\{D_\alpha\}$ of $\operatorname{Der} \wedge\mathcal{E}$ such that

$$\bar{\omega}(D_\alpha, D_\beta) \in \bigoplus_{k\leq 2} \wedge^k \mathcal{E}.$$

In fact, *any* graded symplectic form on $(M, \wedge\mathcal{E})$ is, up to the action of $\operatorname{Aut} \wedge \mathcal{E}$, of this type. The proof of this assertion depends strongly on the fact that $\bar{\omega}$ is closed (see [8] and [11]), and on the fact that the cohomology derived from the graded differential forms is isomorphic to the de Rham cohomology of the base manifold M.

Resuming: Second order depth in the graded Riemannian setting is shown here to be equivalent to $R(H, \cdot\,)H = 0$, where R is the graded curvature of $\boldsymbol{\nabla}$. In the concrete example of the graded manifold defined by the algebra $\Omega(M)$ of differential forms on M, the allowed connection ∇' for which $\nabla'\omega = 0$ has some torsion. It turns out that ω is closed if and only if this torsion belongs to the symplectic algebra of ω. In particular, if ∇' is the Levi-Civita connection of g, then $\mathrm{d}\,\omega = 0$. In other words, we come as close as we can get to the data that defines a graded symplectic structure on a split graded manifold.

Graded Riemannian Curvature. We have computed in §5 the graded curvature and the graded Ricci tensor for the adapted, even, second-order depth metrics. The explicit results show that any notion of 'graded sectional curvature' involving $(2,2)$ dimensional 'planes' would imply that the curvature $R^{\nabla'}(X,Y) \in \operatorname{End}\mathcal{E}^*$—which at each point defines an element of $\mathfrak{sp}_\omega$ in the corresponding fiber—acts like a scalar. Therefore, we restrict the definition of sectional curvature only to nondegenerate $(2,0)$-dimensional submodules of $\operatorname{Der} \wedge\mathcal{E}$. It is then shown (Proposition 5.1) that split graded manifolds $(M, \wedge\mathcal{E})$ of constant curvature can be supported only over an ordinary Riemannian manifold M of constant curvature, and the curvature of the connection ∇' acts as a two-step nilpotent operator: $R^{\nabla'}(X,Y)R^{\nabla'}(X,Y) = 0$. It is also shown (Proposition 5.2) that graded Einstein manifolds can be supported only over Ricci flat manifolds M. Finally (Proposition 5.3) graded Ricci flat manifolds can be supported over Ricci flat manifolds M for which the curvature of ∇' satisfies $\sum_{a,b}(g^{-1})_{ba}R^{\nabla'}(X, X_a)R^{\nabla'}(Y, X_b) = 0$, and

$$\sum_{a,b}(g^{-1})_{ba}(\widetilde{\nabla}_{X_a}R^{\nabla'})(Y, X_b) = 0$$

where $\widetilde{\nabla}$ denotes the connection on the tensor algebra generated by TM, E, T^*M, and E^*, induced by the Levi Civita connection ∇ on TM and the connection ∇' on E.

SPECIAL RESULTS FOR $\Omega(M)$. It is also worth mentioning that among the odd second order depth adapted metrics obtained for the graded manifold defined by the algebra $\Omega(M)$ of differential forms on M, those corresponding to a symmetric tensor κ have been studied before and have been characterized by the condition that the exterior derivative d is a Killing graded vector field (see [9]). But now, the results obtained in §§2–4 below allow us to further explore the role played by the exterior derivative d when a graded adapted metric is given on $\Omega(M)$. Our starting point is the Lie superalgebra $\mathfrak{g}$ of dimension $(1,1)$ generated by H, and d (in this setting, $H = i_{\mathrm{Id}}$, where $\mathrm{Id} \in \Gamma(\mathrm{Hom}(TM, TM)) \simeq \Gamma(T^*M \otimes TM)$):

$$[H,H] = 0, \qquad [H,\mathrm{d}] = \mathrm{d}, \qquad [\mathrm{d},\mathrm{d}] = 0.$$

It is then natural to investigate whether or not $\mathfrak{g}$ generates (local) isometries or (local) conformal transformations for the graded adapted metrics of second order depth. The corresponding problem for symplectic graded geometry has been approached in [13] in connection with BRST quantization of constrained dynamical systems. It has been proved there that $\mathfrak{g}$ does generate graded symplectomorphisms. We have shown here that the answer in the graded Riemannian setting is negative (Proposition 6.1). Another point is to investigate whether or not $\mathbb{V}$ represents $\mathfrak{g}$. That is, whether or not the graded curvature components

$$R(H,\mathrm{d}) = \mathbb{V}_H\mathbb{V}_{\mathrm{d}} - \mathbb{V}_{\mathrm{d}}\mathbb{V}_H - \mathbb{V}_{\mathrm{d}} \qquad \text{and} \qquad R(\mathrm{d},\mathrm{d}) = 2\mathbb{V}_{\mathrm{d}}\mathbb{V}_{\mathrm{d}}$$

vanish identically. One may prove that $\mathbb{V}_{\mathrm{d}}\,\mathrm{d} = 0$ in general for adapted metrics, but the curvature components $R(H,\mathrm{d})$ and $R(\mathrm{d},\mathrm{d})$ are in general nonzero. In fact, for adapted metrics of second order depth, the obstruction for representing $\mathfrak{g}$ by $\mathbb{V}$ is measured by the curvature $R^{\nabla'}$. Besides, the problem of understanding the structure of $\mathbb{V}_{\mathrm{d}}$ is far more complicated than that of $\mathbb{V}_H$. Finally, and inspired by results from [13], [14] and [7], we also investigate whether or not there is a derivation d' such that

$$[H,\mathrm{d}'] = \mathrm{d}' \qquad \text{and} \qquad [\mathrm{d}',\mathrm{d}'] = 0$$

with d' in the $\Omega(M)$-span of H and d and $\langle H,\mathrm{d}'\rangle = 0$. This requires first the existence of a nonvanishing function f on the base manifold M so as to define

$\mathrm{d}' = f\,\mathrm{d} - \mathrm{d}fH$. Then, for even metrics one obtains $\langle H, \mathrm{d}'\rangle = 0$ if and only if $f^{-2}\langle H, H\rangle = cf^{-2}\omega$ is closed (c a constant factor). For odd metrics $\langle H, \mathrm{d}'\rangle = 0$ if and only if K_0 defines a Riemannian metric on M.

1. Graded metrics on $\wedge\mathcal{E}$

Notation and conventions: Let M be a smooth, real, n-dimensional manifold. For any sheaf $\mathcal{F}$ over M we shall freely write $\alpha \in \mathcal{F}$ for a section α of the sheaf $\mathcal{F}$ over an arbitrary open subset on M. If $\mathcal{G}$ is any other sheaf over M, the notation $\Phi: \mathcal{F} \to \mathcal{G}$ shall always be understood as a sheaf morphism, and we shall usually specify its effect on sections by writing $\alpha \mapsto \Phi(\alpha)$. We shall denote by $\wedge\mathcal{E}$ the structure sheaf of a given (split) supermanifold $(M, \wedge\mathcal{E})$. We shall think of $\mathcal{E}$ as the sheaf of sections of a given vector bundle $E \to M$, and $\wedge\mathcal{E}$ is then the sheaf of sections of the exterior algebra bundle $\wedge E \to M$. When we refer to 'a metric on $\wedge\mathcal{E}$' it shall always be understood as an abbreviation for 'a metric on $(M, \wedge\mathcal{E})$'.

We shall now summarize from [9] the pertinent definitions and results needed for this work. For the basics on graded manifolds we refer the reader to [6].

1.1 Definition: A $\mathbb{Z}_2$-graded metric on $\wedge\mathcal{E}$ is a graded-symmetric, nondegenerate $\wedge\mathcal{E}$-bilinear map,

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon \operatorname{Der}\wedge\mathcal{E} \times \operatorname{Der}\wedge\mathcal{E} \to \wedge\mathcal{E}.$$

That is,

(1) $\langle \alpha D_1, D_2\rangle = \alpha\langle D_1, D_2\rangle$, $\alpha \in \wedge\mathcal{E}$,
(2) $\langle D_1, D_2\rangle = (-1)^{|D_1||D_2|}\langle D_2, D_1\rangle$,
(3) The map $D \mapsto \langle D,\,\cdot\,\rangle$ is an isomorphism between the $\wedge\mathcal{E}$-modules $\operatorname{Der}\wedge\mathcal{E}$ and $\operatorname{Hom}(\operatorname{Der}\wedge\mathcal{E}, \wedge\mathcal{E})$,

A graded metric is even (resp., odd) if

$$|\langle D_1, D_2\rangle| + |D_1| + |D_2| \equiv 0 \pmod 2, \qquad (\text{resp.}, \equiv 1 \pmod 2).$$

In any of these cases the graded metric is called homogeneous.

1.2 Definition: A graded connection on $\wedge\mathcal{E}$ is a mapping

$$\nabla\colon \operatorname{Der}\wedge\mathcal{E} \times \operatorname{Der}\wedge\mathcal{E} \to \operatorname{Der}\wedge\mathcal{E},$$
$$(D_1, D_2) \mapsto \nabla_{D_1} D_2$$

satisfying the following conditions:

(1) $\nabla_{D_1}(D_2 + D_3) = \nabla_{D_1} D_2 + \nabla_{D_1} D_3$,
(2) $\nabla_{(D_1+D_2)} D_3 = \nabla_{D_1} D_3 + \nabla_{D_2} D_3$,
(3) $\nabla_{\alpha D_1} D_2 = \alpha \nabla_{D_1} D_2$,
(4) $\nabla_{D_1}(\alpha D_2) = D_1(\alpha) D_2 + (-1)^{|D_1||\alpha|} \alpha \nabla_{D_1} D_2$,

where $\alpha \in \wedge\mathcal{E}$. A graded connection in $\wedge\mathcal{E}$ is called $\mathbf{Z}$-homogeneous of degree $|\nabla|$ if for any pair (D_1, D_2) of homogeneous derivations, $(D_1, D_2; \nabla)$ is homogeneous and

$$|\nabla_{D_1} D_2| = |D_1| + |D_2| + |\nabla|$$

Furthermore, ∇ is said to be **even** (resp., **odd**) if

$$|\nabla_{D_1} D_2| + |D_1| + |D_2| \equiv 0 \pmod 2, \qquad (\text{resp.}, \equiv 1 \pmod 2).$$

1.3 Definition: The **torsion,** T, of a graded connection is defined by

$$T(D_1, D_2) = \nabla_{D_1} D_2 - (-1)^{|D_1||D_2|} \nabla_{D_2} D_1 - [D_1, D_2].$$

1.4 Definition: Let $\langle\ \cdot\ ,\ \cdot\ \rangle$ be a graded metric, and ∇ a graded connection on $\wedge\mathcal{E}$. ∇ is **metric** if, for all homogeneous derivations D, D_1, and D_2,

$$\begin{aligned} D\langle D_1, D_2\rangle = \langle \nabla_D D_1, D_2\rangle &+ (-1)^{|D_1||D|}\langle D_1, \nabla^0_D D_2\rangle \\ &+ (-1)^{|D_1|(|D|+1)}\langle D_1, \nabla^1_D D_2\rangle, \end{aligned}$$

where $\nabla = \nabla^0 + \nabla^1$ is the decomposition of the graded connection into its even and odd components.

1.5 THEOREM: *Given a graded metric, there is a unique torsionless and metric graded connection given by the formula*

$$\begin{aligned} 2\langle \nabla_{D_1} D_2, D_3\rangle = & D_1\langle D_2, D_3\rangle - (-1)^{|D_3|(|D_1|+|D_2|)} D_3\langle D_1, D_2\rangle \\ & + (-1)^{|D_1|(|D_2|+|D_3|)} D_2\langle D_3, D_1\rangle + \langle [D_1, D_2], D_3\rangle \\ & - (-1)^{|D_1|(|D_2|+|D_3|)}\langle [D_2, D_3], D_1\rangle \\ & + (-1)^{|D_3|(|D_1|+|D_2|)}\langle [D_3, D_1], D_2\rangle. \end{aligned}$$

Remark: We shall refer the reader to [9] for the proof. We only remark here that the graded Levi-Civita connection for a homogeneous graded metric is always even. In this work we shall deal exclusively with homogeneous graded metrics.

It is worth mentioning how the group $\mathrm{Aut}\wedge\mathcal{E}$ of $\mathbf{Z}_2$-graded algebra automorphisms of $\wedge\mathcal{E}$ acts on the geometrical objects that we shall be dealing with:

(1) $\mathrm{Aut}\wedge\mathcal{E}$ acts on the left of $\mathrm{Der}\wedge\mathcal{E}$ by graded Lie algebra automorphisms via

$$\Phi\cdot D=\Phi\circ D\circ\Phi^{-1},\qquad D\in\mathrm{Der}\wedge\mathcal{E}.$$

(2) $\mathrm{Aut}\wedge\mathcal{E}$ acts on the right of the graded metrics on $\wedge\mathcal{E}$ by automorphisms via

$$\langle D_1,D_2\rangle^{\Phi}=\Phi^{-1}\langle\Phi\cdot D_1,\Phi\cdot D_2\rangle,\qquad\langle\ \cdot\ ,\ \cdot\ \rangle\text{ a graded metric on }\wedge\mathcal{E}.$$

(3) $\mathrm{Aut}\wedge\mathcal{E}$ acts on the right of the graded connections on $\wedge\mathcal{E}$ by automorphisms via

$$\left(\nabla\cdot\Phi\right)_{D_1}D_2=(\Phi^{-1})\cdot\left(\nabla_{\Phi\cdot D_1}\Phi\cdot D_2\right),\qquad\nabla\text{ a graded connection on }\wedge\mathcal{E}.$$

1.6 Proposition: *Let ∇ (resp., ∇^{Φ}) be the graded Levi-Civita connection of the graded metric $\langle\cdot,\cdot\rangle$ (resp., $\langle\cdot,\cdot\rangle^{\cdot\Phi}$). Then, $\nabla^{\Phi}=\nabla\cdot\Phi$.*

Proof: This is a straightforward computation. ∎

2. Characterization of the graded metrics adapted to the canonical splitting

According to [7], any even connection ∇ on $\wedge\mathcal{E}$ gives rise to a unique splitting derivation $H^{\nabla}\in\mathrm{Der}\wedge\mathcal{E}$ such that $\nabla_{H^{\nabla}}H^{\nabla}=H^{\nabla}$. As it was mentioned before, we shall assume that the splitting H is the canonical splitting that produces the given $\mathbb{Z}$-grading on $\wedge\mathcal{E}=\oplus_{k\geq 0}\wedge^k\mathcal{E}$. We now want to characterize the homogeneous graded metrics on $\wedge\mathcal{E}$ that are adapted to this canonical splitting, i.e., those for which $H^{\nabla}=H$, where ∇ is the corresponding Levi-Civita connection.

2.1 Proposition:

(A) *An even graded metric $\langle\ \cdot\ ,\ \cdot\ \rangle$ is adapted to the canonical splitting if and only if, for any homogeneous derivation D, $\langle H,D\rangle=D\omega_{(2)}$, where $2\omega_{(2)}=\langle H,H\rangle\in\wedge^2\mathcal{E}$.*

(B) *An odd graded metric $\langle\ \cdot\ ,\ \cdot\ \rangle$ is adapted to the canonical splitting if and only if, for any homogeneous derivation D of degree k, $\langle H,D\rangle\in\wedge^{k+1}\mathcal{E}$.*

Proof: Let D be a $\mathbb{Z}$-homogeneous derivation for the $\mathbb{Z}$-grading induced by H. From the definition of the Levi-Civita connection we get

$$2\langle \nabla_H H - H, D\rangle = 2\big(H\langle H, D\rangle - (1+|D|)\langle H, D\rangle\big) - D\langle H, H\rangle.$$

Therefore, H is the canonical splitting for ∇ if and only if

$$2\big(H\langle H, D\rangle - (1+|D|)\langle H, D\rangle\big) = D\langle H, H\rangle.$$

In particular, for $D = H$, obtain $H\langle H, H\rangle = 2\langle H, H\rangle$, and it follows that $\langle H, H\rangle = 2\omega_{(2)} \in \wedge^2\mathcal{E}$. If $\langle\ \cdot\ ,\ \cdot\ \rangle$ is odd, this implies $\langle H, H\rangle = 0$ because $|\langle H, H\rangle|$ should be odd. In short, H is the canonical splitting for ∇ if and only if, for any homogeneous derivation D,

$$H\langle H, D\rangle = \begin{cases} (2+|D|)\langle H, D\rangle \quad \text{and} \quad \langle H, D\rangle = D\omega_{(2)}, & \text{if } \langle\ \cdot\ ,\ \cdot\ \rangle \text{ is even },\\ (1+|D|)\langle H, D\rangle \quad \text{and} \quad \langle H, H\rangle = 0, & \text{if } \langle\ \cdot\ ,\ \cdot\ \rangle \text{ is odd },\end{cases}$$

from which the assertion follows. ∎

Remark: We have used the fact that the $\mathbb{Z}$-grading of $\wedge\mathcal{E}$ is defined by the eigenspaces of the canonical splitting H, and that the $\mathbb{Z}$-grading of $\operatorname{Der}\wedge\mathcal{E}$ is defined by the eigenspaces of $\operatorname{ad}(H)$ (cf. [7]).

This proposition says how $\langle\ \cdot\ ,\ \cdot\ \rangle$ acts on the pairs of derivations (H, D), with D homogeneous. This information, however, does not determine yet the structure of such graded metrics. In order to elucidate its nature even further we need to go into the structure of the left $\wedge\mathcal{E}$-module, $\operatorname{Der}\wedge\mathcal{E}$ (cf. Introduction above):

Use will be made of the fact that $\operatorname{Der}\wedge\mathcal{E}$ is a locally-free sheaf of $\wedge\mathcal{E}$-modules [6], whose structure can be described as follows (see [8] and [11]): Let $\mathcal{E}^*$ be the sheaf of sections of the dual bundle $E^* \to M$. There is a monomorphism $i\colon \mathcal{E}^* \hookrightarrow \operatorname{Der}\wedge\mathcal{E}$ defined by letting each section $\chi \in \mathcal{E}^*$ act on $\wedge\mathcal{E}$ as a degree -1 derivation:

$$i_\chi(s_1\cdots s_k) = \sum_{\alpha\geq 1}(-1)^{\alpha+1}(\chi \mid s_\alpha)s_1\cdots\widehat{s_\alpha}\cdots s_k$$

on decomposable sections $s_1\cdots s_k \in \wedge^k\mathcal{E}$ (with $s_\alpha \in \wedge^1\mathcal{E}$). Similarly, $\operatorname{End}\mathcal{E}$ acts on $\wedge\mathcal{E}$ by degree-preserving derivations by letting $i\colon \operatorname{End}\mathcal{E} \hookrightarrow \operatorname{Der}\wedge\mathcal{E}$ be given by

$$\operatorname{End}\mathcal{E} = \mathcal{E}\otimes\mathcal{E}^* \ni A = \sum_{\mu=1}^{n} s_\mu \otimes \chi_\mu \quad \mapsto \quad i_A = \sum_{\mu=1}^{n} s_\mu i_{\chi_\mu} \in \operatorname{Der}\wedge\mathcal{E}.$$

It easily follows that i extends uniquely to $(\wedge\mathcal{E})\otimes\mathcal{E}^*$, thus giving a monomorphism

$$i\colon (\wedge\mathcal{E})\otimes\mathcal{E}^* \hookrightarrow \operatorname{Der}\wedge\mathcal{E}.$$

In particular, $\mathrm{Id}\in\operatorname{End}\mathcal{E}$, and $H = i_{\mathrm{Id}}$ is the canonical splitting of $\wedge\mathcal{E}$. On the other hand, let $\mathfrak{X}_M = \operatorname{Der} C^\infty_M$ be the sheaf of smooth vector fields on M. A connection ∇ on $\wedge\mathcal{E}$ gives, by definition, a morphism

$$\begin{gathered}(\wedge\mathcal{E})\otimes\mathfrak{X}_M \to \operatorname{Der}\wedge\mathcal{E}\\ \alpha\otimes X \mapsto \alpha\nabla_X\end{gathered}$$

which is in fact a splitting for the exact sequence of locally free sheaves of $\wedge\mathcal{E}$-modules

$$0\to\wedge\mathcal{E}\otimes\mathcal{E}^*\to\operatorname{Der}\wedge\mathcal{E}\to\wedge\mathcal{E}\otimes\mathfrak{X}_M\to 0.$$

The projection $\operatorname{Der}\wedge\mathcal{E}\to\wedge\mathcal{E}\otimes\mathfrak{X}_M$ is given on a filtration degree k derivation $D\in\operatorname{Der}\wedge\mathcal{E}$ as follows:

$$\begin{gathered}D\mapsto\tilde D\in\operatorname{Hom}\bigl((\wedge\mathcal{E})^*,\mathfrak{X}_M\bigr)\simeq\operatorname{Hom}\bigl(\wedge\mathcal{E}^*,\mathfrak{X}_M\bigr)\simeq\wedge\mathcal{E}\otimes\mathfrak{X}_M,\\ \tilde D(\chi_1\cdots\chi_k) = i_{\chi_1}\cdots i_{\chi_k}D.\end{gathered}$$

We shall fix for the moment a linear connection ∇ on $\wedge E$. Then there is a set of sections associated to a given graded metric $\langle\,\cdot\,,\,\cdot\,\rangle$ that can be described in terms of the basic derivations ∇_X, and i_χ ($X\in\mathfrak{X}_M$ and $\chi\in\mathcal{E}^*$): The map $(\chi,\varphi)\mapsto\langle i_\chi, i_\varphi\rangle\in\wedge\mathcal{E}$ is clearly $C^\infty(M)$-linear and skew-symmetric. It therefore defines a unique element $L\in\wedge^2\mathcal{E}\otimes\wedge\mathcal{E}$. We shall write

$$\langle i_\chi, i_\varphi\rangle = L_0(\chi,\varphi) + L_1(\_;\chi,\varphi) + L_2(\_,\_;\chi,\varphi)+\cdots,\quad L_k\in\wedge^2\mathcal{E}\otimes\wedge^k\mathcal{E}.$$

We shall also write

$$\langle i_\chi, i_\varphi\rangle = L_0(\chi,\varphi) + L_1(\chi,\varphi) + L_2(\chi,\varphi)+\cdots,\quad L_k(\chi,\varphi)\in\wedge^k\mathcal{E}.$$

In a similar manner, the map $(X,Y)\mapsto\langle\nabla_X,\nabla_Y\rangle\in\wedge\mathcal{E}$, being $C^\infty(M)$-linear and symmetric, defines a unique element $P\in S^2\mathfrak{X}^*_M\otimes\wedge\mathcal{E}$. We shall write

$$\langle\nabla_X,\nabla_Y\rangle = P_0(X,Y) + P_1(\_;X,Y) + P_2(\_,\_;X,Y)+\cdots,$$

where $P_k\in S^2\mathfrak{X}^*_M\otimes\wedge^k\mathcal{E}$. Finally, $(X,\chi)\mapsto\langle\nabla_X, i_\chi\rangle$ defines a unique element $K\in\mathfrak{X}^*_M\otimes\mathcal{E}\otimes\wedge\mathcal{E}$, and

$$\langle\nabla_X, i_\chi\rangle = K_0(X,\chi) + K_1(\_;X,\chi) + K_2(\_,\_;X,\chi)+\cdots,$$

with $K_k\in\mathfrak{X}^*_M\otimes\mathcal{E}\otimes\wedge^k\mathcal{E}$. We summarize this as follows:

2.2 Proposition: *Let ∇ be a linear connection on $\wedge E$, and let $\eta \in \mathbb{Z}_2$. There is a one-to-one correspondence between the set of graded metrics of degree η on $\wedge\mathcal{E}$, and the set of sections (P, K, L), with*

$$\begin{aligned}
P &= \sum_{k\geq 0} P_{2k+\eta} \in \bigoplus_{k\geq 0} S^2\mathfrak{X}_M^* \otimes \wedge^{2k+\eta}\mathcal{E},\\
K &= \sum_{k\geq 0} K_{2k+1-\eta} \in \bigoplus_{k\geq 0} \mathfrak{X}_M^* \otimes \mathcal{E} \otimes \wedge^{2k+1-\eta}\mathcal{E},\\
L &= \sum_{k\geq 0} L_{2k+\eta} \in \bigoplus_{k\geq 0} \wedge^2\mathcal{E} \otimes \wedge^{2k+\eta}\mathcal{E},
\end{aligned}$$

where, in the even case, P_0 and L_0 are non-degenerate, and in the odd case K_0 is non-degenerate.

Remark: When $\langle\cdot\,,\cdot\rangle$ is even, P_0 and L_0 must be nondegenerate. In particular, $P_0 = g$ defines a pseudo-Riemannian metric on M, and $L_0 = \omega$ a nondegenerate skew-symmetric fiber bundle map $E^* \times E^* \to \mathbb{R}$; the latter restricts the rank of E to be even dimensional. On the other hand, when $\langle\cdot\,,\cdot\rangle$ is odd, $K_0 = \kappa$ must be a nondegenerate bilinear pairing between $\mathfrak{X}_M$ and $\mathcal{E}^*$, thus restricting the supermanifold dimension to (n, n).

Convention: We shall write a homogeneous graded metric $\langle\cdot\,,\cdot\rangle$ on $\wedge\mathcal{E}$ in the form,

$$\langle\cdot,\cdot\rangle = \begin{cases} \begin{pmatrix} g+P & K \\ K^t & \omega+L \end{pmatrix} \quad \text{where} \quad \left.\begin{aligned} P &= P_2+P_4+\cdots \\ K &= K_1+K_3+\cdots \\ L &= L_2+L_4+\cdots \end{aligned}\right\} & \text{(even metric)} \\[2ex] \begin{pmatrix} P & \kappa+K \\ \kappa^t+K^t & L \end{pmatrix} \quad \text{where} \quad \left.\begin{aligned} P &= P_1+P_3+\cdots \\ K &= K_2+K_4+\cdots \\ L &= L_1+L_3+\cdots \end{aligned}\right\} & \text{(odd metric)} \end{cases}$$

where $P_k \in \wedge^k\mathcal{E} \otimes S^2(\mathfrak{X}_M^*)$, $L_k \in \wedge^k\mathcal{E} \otimes \wedge^2\mathcal{E}$, and $K_k \in \wedge^k\mathcal{E} \otimes \mathfrak{X}_M^* \otimes \mathcal{E}$ are defined in terms of a given connection ∇. These matrices may be understood as $\wedge\mathcal{E}$-valued matrices, provided that local frames $\{X_a\}$ on M and $\{\chi_\alpha\}$ on E^* have been chosen, and we write

$$K_{a\beta} = K_0(X_a, \chi_\beta) + K_1(\_; X_a, \chi_\beta) + K_2(\_, \_; X_a, \chi_\beta) + \cdots,$$

with similar conventions for P_{ab} and $L_{\alpha\beta}$.

We shall now throw in the information from Proposition 2.1 of how $\langle H, \cdot \rangle$ acts on the basic homogeneous derivations ∇_X, and i_χ ($X \in \mathfrak{X}_M$, and $\chi \in \mathcal{E}^*$), when $\boldsymbol{\nabla}_H H = H$, to further restrict the possibilities for the sections P, K, and L. We shall make use of the following Young symmetrizers (cf. Proposition 2.3 below). First, let

$$A\colon \wedge^2 \mathcal{E} \otimes \wedge^k \mathcal{E} \to \wedge^{k+2}\mathcal{E} = (\wedge^{k+2}\mathcal{E}^*)^* = \mathrm{Hom}(\wedge^{k+2}\mathcal{E}^*, \wedge^0\mathcal{E})$$

be the map defined by

$$(AL)(\chi_1, \dots, \chi_{k+2}) = \sum_{\alpha=1}^{k+1}(-1)^{\alpha-1}L(\chi_1, \dots, \widehat{\chi_\alpha}, \dots, \chi_{k+1}; \chi_\alpha, \chi_{k+2}).$$

It is immediate to verify that A is well defined, and $A^2 = A$ under the appropriate identifications. Note that if $L = L_2$,

$$AL_2(\chi_1, \chi_2; \chi_3, \chi_4) = \sum_{\text{Cyclic } \{1,2,3\}} L_2(\chi_1, \chi_2; \chi_3, \chi_4).$$

In particular, $\mathrm{Ker}(A|_{\Gamma(\wedge^2 T^* M \otimes \wedge^2 T^* M)})$ has the same symmetries as the **algebraic curvature tensors**, since $L_2(\chi_1, \chi_2; \chi_3, \chi_4) + L_2(\chi_1, \chi_2; \chi_4, \chi_3) = 0$, and $L_2(\chi_1, \chi_2; \chi_3, \chi_4) + L_2(\chi_2, \chi_1; \chi_3, \chi_4) = 0$ ([2] Dfn. 1.108). Now let

$$B\colon \wedge^k \mathcal{E} \otimes \mathfrak{X}_M^* \otimes \mathcal{E} \to \wedge^{k+1}\mathcal{E} \otimes \mathfrak{X}_M^* \simeq \mathrm{Hom}((\wedge^{k+1}\mathcal{E}^*) \otimes \mathfrak{X}_M, \wedge^0\mathcal{E})$$

be the map defined by

$$(BK)(\chi_1, \dots, \chi_{k+1}; X) = \sum_{\alpha=1}^{k+1}(-1)^{\alpha-1}K(\chi_1, \dots, \widehat{\chi_\alpha}, \dots, \chi_{k+1}; X, \chi_\alpha).$$

Again, B is well defined and $B^2 = B$ under the appropriate identifications. In particular, if $K = K_2$,

$$BK_2(\chi_1, \chi_2, \chi_3; X) = \sum_{\text{Cyclic } \{1,2,3\}} K_2(\chi_1, \chi_2; X, \chi_3),$$

whereas if $K = K_3$,

$$\begin{aligned} BK_3(\chi_1, \chi_2, \chi_3, \chi_4, X) = {} & K_3(\chi_2, \chi_3, \chi_4; X, \chi_1) - K_3(\chi_1, \chi_3, \chi_4; X, \chi_2) \\ & + K_3(\chi_1, \chi_2, \chi_4; X, \chi_3) - K_3(\chi_1, \chi_2, \chi_3; X, \chi_4). \end{aligned}$$

2.3 Proposition: *Let ∇ be a linear connection on M.*

(A) *An even graded metric $\langle \cdot , \cdot \rangle$ is adapted to the canonical splitting H if and only if*

$$\begin{cases} \langle i_\chi, i_\varphi \rangle = -\omega_{(2)}(\chi, \varphi) + L_+(\_;\chi,\varphi), \\ \langle \nabla_X, i_\chi \rangle = K_1(\_;X,\chi) + K_+(\_;X,\chi), \end{cases}$$

where

$$L_+ \in \Big(\bigoplus_{k\geq 1} \wedge^{2k}\mathcal{E} \otimes \wedge^2\mathcal{E}\Big) \cap \operatorname{Ker} A, \qquad K_+ \in \Big(\bigoplus_{k>1} \wedge^{2k-1}\mathcal{E} \otimes \mathcal{E} \otimes \mathfrak{X}_M^*\Big) \cap \operatorname{Ker} B$$

and

$$K_1(\chi;Y,\varphi) - K_1(\varphi;Y,\chi) = (\nabla_Y \omega_{(2)})(\varphi,\chi).$$

(B) *An odd graded metric $\langle \cdot , \cdot \rangle$ is adapted to the canonical splitting H if and only if*

$$\begin{cases} \langle i_\chi, i_\varphi \rangle = L(\chi,\varphi), \\ \langle \nabla_X, i_\chi \rangle = K_0(X,\chi) + K_+(X;\chi), \end{cases}$$

where

$$L \in \Big(\bigoplus_{k\geq 1} \wedge^{2k-1}\mathcal{E} \otimes \wedge^2\mathcal{E}\Big) \cap \operatorname{Ker} A; \qquad K_+ \in \Big(\bigoplus_{k\geq 1} \wedge^{2k}\mathcal{E} \otimes \mathfrak{X}_M^* \otimes \mathcal{E}\Big) \cap \operatorname{Ker} B.$$

Proof: Let $\{X_a\}$ be a local basis of vector fields and let $\{s_\alpha\}$ and $\{\chi_\beta\}$ be local dual bases for sections of $\mathcal{E}$ and $\mathcal{E}^*$, respectively. The identity $\mathrm{Id} \in \operatorname{End}\mathcal{E}$ can be locally written as $\sum s_\alpha \otimes \chi_\alpha$. Let us suposse that $\langle i_\chi, i_\varphi\rangle = L(\chi,\varphi)$ where $L \in \bigoplus_{k\geq 0} \wedge^{2k}\mathcal{E} \otimes \wedge^2 E$. From the equality $\langle H, i_\varphi\rangle = i_\varphi\omega_{(2)}$ we obtain

$$\sum_\alpha s_\alpha L(\chi_\alpha,\varphi) = i_\varphi \omega_{(2)}.$$

From this we have $L_{(0)} = -\omega_{(2)}$, and $L_+ = L - L_{(0)} \in \operatorname{Ker} A$. The other proofs are similar. ∎

Remark: So far, we have used a connection ∇ on $\wedge E$ to define the tensors P, K, and L associated to a graded metric. For example, L is clearly independent of the choice of ∇, since it is defined through $\langle i_\chi, i_\varphi\rangle$. The question of how K depends on ∇ is easily settled directly from the definitions and the previous results. We may write $\nabla'_Y - \nabla_Y = -i_{A(Y)}$ as derivations of $\wedge\mathcal{E}$, with $A(Y) \in \operatorname{End}\wedge\mathcal{E}$. Then,

$$K_1^{\nabla'}(\chi; Y, \varphi) = K_1^{\nabla}(\chi; Y, \varphi) - (i_{A(Y)}\omega_{(2)})(\chi, \varphi)$$

and

$$K_+^{\nabla'}(\_; Y, \varphi) = K_+^{\nabla}(\_; Y, \varphi) + L_+(\_; A(Y)\_, \varphi).$$

We shall see in §4 below how, under certain hypotheses, a natural change of connection exists that also simplifies the P-entries.

We conclude this section with the following result which is a consequence of the previous proposition, and of proposition 1.2 of [7]:

2.4 THEOREM: *Let $\langle\cdot\,,\,\cdot\rangle$ be a homogeneous graded metric; then there exists an automorphism of the graded algebra of differential forms, Φ: $\wedge\mathcal{E} \to \wedge\mathcal{E}$, such that the metric $\langle\cdot\,,\,\cdot\rangle^{\Phi}$ has the form described in the previous proposition.*

Proof: Let G be a homogeneous graded metric, let $\mathbb{\nabla}^G$ be its Levi-Civita connection and let $H^{\mathbb{\nabla}^G}$ be its adapted derivation. Proposition 1.2 of Koszul says that the group of automorphisms of $\wedge\mathcal{E}$ acts transitively on the set of adapted derivations. Then, there exists an automorphism Φ that transforms $H^{\mathbb{\nabla}^G}$ into H. Therefore, the transformed graded metric $\langle\cdot\,,\,\cdot\rangle^{\Phi}$ is adapted to the canonical splitting. ∎

Remark: This result limits further reductions performed on a graded metric via automorphisms of $\wedge\mathcal{E}$. In fact, based on the grounds that $\mathbb{\nabla}_H H = H$, any automorphism must preserve H. That is, $\operatorname{Ad}(\Phi)H = H$, which amounts to determining all derivations D with $\operatorname{ad}(D)H = 0$, but this is precisely the set of degree-zero derivations. Hence, Φ can be any automorphism of E. This situation is to be contrasted with the graded symplectic case; the fact of having the pairing $\langle\cdot\,,\,\cdot\rangle$ defined by a **closed** nondegenerate graded 2-form is strong enough so as to use the automorphism group $\operatorname{Aut}\wedge\mathcal{E}$ via graded symplectomorphisms generated by degree-increasing derivations, which can get rid of most of the tensors appearing in $\langle\cdot\,,\,\cdot\rangle$. In the even case, one ends up with the data $\{\omega, g, \nabla\}$, whereas in the odd case, with $\{\kappa, \nabla\}$ (cf. [8], and [11]). The purpose of the next section is to determine a condition under which a similar situation can be attained for graded metrics.

3. A condition on the graded curvature

Let R be the graded curvature of $\mathbb{V}$,

$$R(D_1, D_2)D_3 = \mathbb{V}_{D_1}\mathbb{V}_{D_2}D_3 - (-1)^{|D_1||D_2|}\mathbb{V}_{D_2}\mathbb{V}_{D_1}D_3 - \mathbb{V}_{[D_1,D_2]}D_3.$$

We shall study the endomorphism

$$\operatorname{Der}\wedge\mathcal{E} \ni D \mapsto \rho(D) = R(H,D)H \in \operatorname{Der}\wedge\mathcal{E}.$$

Using the fact that $\mathbb{V}$ is torsionless, and the fact that H is the canonical splitting of $\mathbb{V}$, we get

$$\rho(D) = \mathbb{V}_H\Big(\mathbb{V}_H D - [H,D]\Big) - \Big(\mathbb{V}_H D - [H,D]\Big) - \Big(\mathbb{V}_H[H,D] - [H,[H,D]]\Big).$$

In particular, for a $\mathbb{Z}$-homogeneous derivation D, of degree $|D|$,

$$\rho(D) = \mathbb{V}_H\Big(\mathbb{V}_H D - |D|D\Big) - \Big(\mathbb{V}_H D - |D|D\Big) - |D|\Big(\mathbb{V}_H D - |D|D\Big).$$

It follows that

$$\rho(D) = \mathbb{V}_H\Big(\mathbb{V}_H D - (-1)^{|D|}D\Big),$$

or

$$\rho = \mathbb{V}_H \circ \mathbb{V}_H - \mathbb{V}_H \circ \Gamma$$

where Γ: $\operatorname{Der}\wedge\mathcal{E} \to \operatorname{Der}\wedge\mathcal{E}$ is the $\mathbb{R}$-linear map whose value on a homogeneous derivation D of degree $|D|$ is $\Gamma(D) = (-1)^{|D|}D$.

It is the purpose of this section to show that, under the additional geometric assumption that ρ be identically zero, one may cut down the number of tensors appearing on a homogeneous graded metric adapted to the canonical splitting. More precisely, we shall prove the following result:

3.1 PROPOSITION: *Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a homogeneous graded metric whose Levi-Civita connection is adapted to the canonical splitting H. Then, the following conditions are equivalent:*

(1) *$\langle\,\cdot\,,\,\cdot\,\rangle$ has at most* **second order depth**; *that is,*

$$\langle\,\cdot\,,\,\cdot\,\rangle = \begin{cases} \begin{pmatrix} g+P_2 & K_1 \\ K_1^t & \omega \end{pmatrix} & \text{with} \quad P_2 - K_1\omega^{-1}K_1^t = 0 \quad \textit{(even metric)}, \\ \begin{pmatrix} P_1 & \kappa \\ \kappa^t & 0 \end{pmatrix} & \qquad\qquad\qquad\qquad \textit{(odd metric)}. \end{cases}$$

(2) $\mathbb{V}_H\nabla_X \in \operatorname{Span}_{\wedge^1\mathcal{E}}\{i_X\}$, *and* $\mathbb{V}_H i_X = 0$.

(3) $\rho(D) = 0$, *for all* $D \in \operatorname{Der}\wedge\mathcal{E}$.

The proof will follow from Lemmas 3.2–3.5 below, each of which has a separate interest. The first two lemmas have to do with the structure of a general homogeneous graded metric and that of its inverse. The third lemma gives the structure of $\mathbb{V}_H$ in terms of $\{\nabla_X, i_X\}$ under the assumption that $\mathbb{V}_H H = H$, and finally, ρ is explicitly computed from the known structure of $\mathbb{V}_H$ in Lemma 3.5. Throughout this section, an implicit use of a fixed connection ∇ will be made.

GENERAL STRUCTURE OF $\langle\cdot,\cdot\rangle$. We shall describe the relationship between the $\mathbb{Z}$-graded structure of the most general homogeneous graded metric and the $\mathbb{Z}$-graded structure of its inverse. We shall write

$$\langle\cdot,\cdot\rangle = \widetilde{G} + \zeta, \quad \text{where} \quad \zeta = \sum_{k\geq 1}\zeta_k \quad \text{and} \quad \widetilde{G} = \begin{cases} \begin{pmatrix} g & 0 \\ 0 & \omega \end{pmatrix} & \text{(even)}, \\ \begin{pmatrix} 0 & \kappa \\ \kappa^t & 0 \end{pmatrix} & \text{(odd)}. \end{cases}$$

Here, ζ_k is a matrix with entries in $\wedge^k\mathcal{E}$. In fact,

$$\zeta_{2k-1} = \begin{cases} \begin{pmatrix} 0 & K_{2k-1} \\ K^t_{2k-1} & 0 \end{pmatrix} \\ \begin{pmatrix} P_{2k-1} & 0 \\ 0 & L_{2k-1} \end{pmatrix} \end{cases} \quad \text{and } \zeta_{2k} = \begin{cases} \begin{pmatrix} P_{2k} & 0 \\ 0 & L_{2k} \end{pmatrix} & \text{(even metric)}, \\ \begin{pmatrix} 0 & K_{2k} \\ K^t_{2k} & 0 \end{pmatrix} & \text{(odd metric)}. \end{cases}$$

Now, the inverse of $\langle\cdot,\cdot\rangle$ can be explicitly computed by

$$\langle\cdot,\cdot\rangle^{-1} = \sum_{j\geq 0}(-1)^j(\widetilde{G}^{-1}\zeta)^j\widetilde{G}^{-1}.$$

It is convenient to write this (finite) series as an expansion ordered by the $\mathbb{Z}$-grading. Namely,

$$\langle\cdot,\cdot\rangle^{-1} = -\sum_{k\geq 0}\widetilde{G}^{-1}\xi_k\widetilde{G}^{-1}$$

where ξ_k is a matrix with entries in $\wedge^k\mathcal{E}$, and

$$\xi_0 = -\widetilde{G} \quad \text{by convention,} \qquad \text{and} \qquad \xi_1 = \zeta_1 \quad \text{by definition.}$$

3.2 LEMMA: *Let the notation be as above. Denote by $\zeta^{(j)}_k$ the terms lying in $\wedge^k\mathcal{E}$ coming from $\widetilde{G}(\widetilde{G}^{-1}\zeta)^j$ (this makes sense for $1 \leq j \leq k$, and clearly $\zeta^{(1)}_k = \zeta_k$). Then,*

$$\zeta^{(j)}_k = \sum_{i=1}^{k-j+1}\zeta^{(j-1)}_{k-i}\widetilde{G}^{-1}\zeta_i, \qquad 1 \leq j \leq k,$$

and furthermore,

$$\xi_k = \sum_{j=1}^{k} (-1)^{j-1} \zeta_k^{(j)}.$$

Proof: This is a straightforward proof by induction. ∎

We now want to investigate the conditions under which the assumption $\zeta_3 = 0$ leads to $\xi_3 = 0$, and also try to tie them up with the conditions under which $\zeta_4 = 0$ leads to $\xi_4 = 0$, and so on. For this reason, the important terms to look at will be ξ_2, ξ_3, and ξ_4. It is easy to check that

$$\begin{aligned}
\xi_2 &= \zeta_2 - \zeta_1 \widetilde{G}^{-1}\zeta_1,\\
\xi_3 &= \zeta_3 - \zeta_1 \widetilde{G}^{-1}\xi_2 - \zeta_2 \widetilde{G}^{-1}\zeta_1,\\
\xi_4 &= \zeta_4 + (\zeta_3^{(2)} - \zeta_3^{(3)})\widetilde{G}^{-1}\zeta_1 - \xi_2 \widetilde{G}^{-1}\zeta_2 - (\zeta_3 \widetilde{G}^{-1}\zeta_1 + \zeta_1 \widetilde{G}^{-1}\zeta_3).
\end{aligned}$$

Note that

$$\xi_2 = \begin{cases} \begin{pmatrix} P_2 - K_1\omega^{-1}K_1^t & 0 \\ 0 & L_2 - K_1^t g^{-1} K_1 \end{pmatrix} & \text{(even metric)}, \\[2ex] \begin{pmatrix} 0 & K_2 - P_1(\kappa^{-1})^t L_1 \\ K_2^t - L_1\kappa^{-1}P_1 & 0 \end{pmatrix} & \text{(odd metric)}. \end{cases}$$

Also note that the difference $\zeta_3^{(2)} - \zeta_3^{(3)} = -(\xi_3 - \zeta_3)$ is given by

$$\zeta_3^{(2)} - \zeta_3^{(3)} = \begin{cases} \begin{pmatrix} 0 & \begin{matrix}(P_2 - K_1\omega^{-1}K_1^t)g^{-1}K_1 \\ +K_1\omega^{-1}L_2\end{matrix} \\ \begin{matrix}K_1^t g^{-1}(P_2 - K_1\omega^{-1}K_1^t) \\ +L_2\omega^{-1}K_1^t\end{matrix} & 0 \end{pmatrix} \\ \qquad\qquad \text{(even metric)} \\[2ex] \begin{pmatrix} \begin{matrix}K_2\kappa^{-1}P_1 + P_1(\kappa^{-1})^t K_2^t \\ -P_1(\kappa^{-1})^t L_1 \kappa^{-1} P_1\end{matrix} & 0 \\ 0 & \begin{matrix}K_2^t(\kappa^{-1})^t L_1 + L_1\kappa^{-1}K_2 \\ -L_1\kappa^{-1}P_1(\kappa^{-1})^t P_1\end{matrix} \end{pmatrix} \\ \qquad\qquad \text{(odd metric)} \end{cases}$$

and, finally,

$$\xi_2 \widetilde{G}^{-1}\zeta_2 = \begin{cases} \begin{pmatrix} (P_2 - K_1\omega^{-1}K_1^t)g^{-1}P_2 & 0 \\ 0 & (L_2 - K_1^t g^{-1}K_1)\omega^{-1}L_2 \end{pmatrix} \\ \qquad\qquad \text{(even metric)}, \\[2ex] \begin{pmatrix} 0 & (K_2 - P_1(\kappa^{-1})^t L_1)\kappa^{-1}K_2^t \\ (K_2^t - L_1\kappa^{-1}P_1)(\kappa^{-1})^t K_2 & 0 \end{pmatrix} \\ \qquad\qquad \text{(odd metric)}. \end{cases}$$

3.3 LEMMA:

(A) *Let $\langle\ \cdot,\cdot\rangle$ be a homogeneous graded metric. Then $\zeta_3^{(2)}-\zeta_3^{(3)}$ and $\xi_2\widetilde{G}^{-1}\zeta_2$ vanish identically if either*

$$\begin{cases} P_2-K_1\omega^{-1}K_1^t=0 \text{ and } L_2=0, & \text{or}\\ L_2-K_1^t g^{-1}K_1=0 \text{ and } L_2=0, \end{cases}$$

when the metric is even, and if either

$$\begin{cases} P_1=0, \ \text{and} \ K_2=0, & \text{or}\\ L_1=0, \ \text{and} \ K_2=0, \end{cases}$$

if the metric is odd.

(B) *Suppose $\zeta_3^{(2)}-\zeta_3^{(3)}$ and $\xi_2\widetilde{G}^{-1}\zeta_2$ vanish identically, and assume that for some integer $k\geq 4$, $\zeta_3=\cdots=\zeta_{k-1}=0$. Then, $\xi_3=\cdots=\xi_{k-1}=0$. Moreover,*

$$\xi_k=\zeta_k \qquad \text{and} \qquad \xi_{k+1}=\zeta_{k+1}-\zeta_1\widetilde{G}^{-1}\zeta_k-\zeta_k\widetilde{G}^{-1}\zeta_1.$$

Proof: The proof is by induction. The case $k=4$ follows from the explicit expressions above. Now let $k>4$ be as in the statement. The induction hypothesis (applied to $k-1$) says that when $\zeta_3=\cdots=\zeta_{k-2}=0$, then $\xi_3=\cdots=\xi_{k-2}=0$, and

$$\xi_{k-1}=\zeta_{k-1} \qquad \text{and} \qquad \xi_k=\zeta_k-\zeta_1\widetilde{G}^{-1}\zeta_{k-1}-\zeta_{k-1}\widetilde{G}^{-1}\zeta_1.$$

But then, if k is as in the statement, $\zeta_{k-1}=0$ and hence $\xi_{k-1}=0$, and $\xi_k=\zeta_k$. The expression for ξ_{k+1} follows now from the general formula for ξ_k given in the previous lemma, and the induction hypothesis. ∎

THE STRUCTURE OF $\mathbb{V}_H D$. We now want to compute $\mathbb{V}_H D$ for D ranging over a basis of graded derivations, say $\{\nabla_{X_a}, i_{\chi_\alpha}\}$, where $\{X_a\}$ is a local frame on M, and $\{\chi_\alpha\}$ is a frame for $\mathcal{E}^*$ dual to a frame $\{s_\alpha\}$ of $\mathcal{E}$.

The starting point is the formula for the Levi-Civita connection. Using it in conjunction with Proposition 1 of §2, one obtains

$$\langle\mathbb{V}_H D_1,D_2\rangle=\begin{cases} \frac{1}{2}\Big(H\langle D_1,D_2\rangle+(|D_1|-|D_2|)\langle D_1,D_2\rangle\Big) \\ \qquad\qquad \text{(even metric)}, \\ \\ \frac{1}{2}\Big(H\langle D_1,D_2\rangle+(|D_1|-|D_2|)\langle D_1,D_2\rangle\Big) \\ \quad +\Big(D_1\langle H,D_2\rangle-(-1)^{|D_1||D_2|}D_2\langle H,D_1\rangle-\langle H,[D_1,D_2]\rangle\Big) \\ \qquad\qquad \text{(odd metric)}. \end{cases}$$

Even though the formula for the odd metric looks more complicated, it may be simplified enormously when D_1 and D_2 run over a set of generators —say $\{\nabla_X,\ i_\chi\}$—since the odd adapted metrics satisfy

$$|D| = -1 \implies \langle H, D\rangle = 0 \qquad \text{and} \qquad |D| = 0 \implies \langle H, D\rangle \in \wedge^1\mathcal{E}.$$

In fact, it is a straightforward matter to verify that

$$\begin{pmatrix} \langle \mathbb{V}_H\nabla_X, \nabla_Y\rangle & \langle \mathbb{V}_H\nabla_X, i_\varphi\rangle \\ \langle \mathbb{V}_H i_\chi, \nabla_Y\rangle & \langle \mathbb{V}_H i_\chi, i_\varphi\rangle \end{pmatrix} = \frac{H}{2}\begin{pmatrix} \langle \nabla_X, \nabla_Y\rangle & \langle \nabla_X, i_\varphi\rangle \\ \langle i_\chi, \nabla_Y\rangle & \langle i_\chi, i_\varphi\rangle \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 0 & \langle \nabla_X, i_\varphi\rangle \\ -\langle i_\chi, \nabla_Y\rangle & 0 \end{pmatrix} + \frac{|\langle\cdot,\cdot\rangle|}{2}\begin{pmatrix} (\nabla_X\kappa)(Y,\_) - (\nabla_Y\kappa)(X,\_) & -\kappa(X,\varphi) \\ \kappa(Y,\chi) & 0 \end{pmatrix}$$

where $|\langle\cdot,\cdot\rangle|$ is the $\mathbb{Z}_2$-degree of the metric (i.e., 0 or 1). We may also write a $\mathbb{Z}$-expansion for the matrix $\langle \mathbb{V}_H\cdot,\cdot\rangle$ in terms of the frame $\{\nabla_{X_a}, i_{\chi_\alpha}\}$ as follows:

$$\langle \mathbb{V}_H\cdot,\cdot\rangle = \sum_{k\geq 1}\eta_k$$

where η_k is a matrix with entries in $\wedge^k\mathcal{E}$, and it is explicitly given in terms of the graded metric sections by

$$\eta_{2j-1} = \begin{cases} \begin{pmatrix} 0 & jK_{2j-1} \\ (j-1)K^t_{2j-1} & 0 \end{pmatrix}, & j\geq 1 \qquad \text{(even metric)} \\ \frac{1}{2}\begin{pmatrix} (2j-1)P_{2j-1} & 0 \\ 0 & (2j-1)L_{2j-1} \end{pmatrix}, & j\geq 2 \qquad \text{(even metric)} \end{cases}$$

with

$$\eta_1 = \frac{1}{2}\begin{pmatrix} P_1 + \delta\kappa & 0 \\ 0 & L_1 \end{pmatrix},\ \delta\kappa(X,Y) = (\nabla_X\kappa)(Y,\_) - (\nabla_Y\kappa)(X,\_) \ \text{(odd metric)}$$

and

$$\eta_{2j} = \begin{cases} \begin{pmatrix} jP_{2j} & 0 \\ 0 & jL_{2j} \end{pmatrix}, & j\geq 1 \qquad \text{(even metric),} \\ \frac{1}{2}\begin{pmatrix} 0 & (2j+1)K_{2j} \\ (2j-1)K^t_{2j} & 0 \end{pmatrix}, & j\geq 1 \qquad \text{(odd metric).} \end{cases}$$

Note in particular that $\eta_k = 0$ *if and only if* $\zeta_k = 0$. Now write

$$\mathbb{V}_H\nabla_{X_a} = \sum_{b\geq 1} A_{ab}\nabla_{X_b} + \sum_{\beta\geq 1} B_{a\beta} i_{\chi_\beta},$$

$$\mathbb{V}_H i_{\chi_\alpha} = \sum_{b\geq 1} C_{\alpha b}\nabla_{X_b} + \sum_{\beta\geq 1} D_{\alpha\beta} i_{\chi_\beta},$$

so that the matrix coefficients A, B, C, and D are given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\Big(\sum_{k\geq 1} \eta_k\Big)\Big(\sum_{l\geq 0} \widetilde{G}^{-1}\xi_l\widetilde{G}^{-1}\Big).$$

This may be arranged so that the expansion becomes ordered by $\mathbb{Z}$-degrees, namely,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sum_{k\geq 1}\theta_k = -\sum_{k\geq 1}\Big(\sum_{l=1}^{k} \eta_l\widetilde{G}^{-1}\xi_{k-l}\widetilde{G}^{-1}\Big).$$

Note that the expansion has no matrix with entries in $\wedge^0\mathcal{E} \simeq C^\infty(M)$. Also note that

$$\theta_1 = \begin{cases} \begin{pmatrix} 0 & K_1\omega^{-1} \\ 0 & 0 \end{pmatrix} & \text{(even metric)} \\ \frac{1}{2}\begin{pmatrix} 0 & (P_1+\delta\kappa)(\kappa^{-1})^t \\ L_1\kappa^{-1} & 0 \end{pmatrix} & \text{(odd metric)} \end{cases}$$

and

$$\theta_2 = \begin{cases} \begin{pmatrix} (P_2 - K_1\omega^{-1}K_1^t)g^{-1} & 0 \\ 0 & L_2\omega^{-1} \end{pmatrix} & \text{(even metric)}, \\ \frac{1}{2}\begin{pmatrix} 3K_2\kappa^{-1} - (P_1+\delta\kappa)(\kappa^{-1})^tL_1\kappa^{-1} & 0 \\ 0 & K_2^t(\kappa^{-1})^t - L_1\kappa^{-1}P_1(\kappa^{-1})^t \end{pmatrix} \\ \qquad\qquad \text{(odd metric)}. \end{cases}$$

This time we want to impose conditions on the graded metric tensors, in order to conclude that $\theta_3 = 0$ because $\zeta_3 = 0$, and $\theta_3 = \theta_4 = 0$ because $\zeta_3 = \zeta_4 = 0$, etc. Note that

$$\theta_3 = \eta_3\widetilde{G}^{-1} - \eta_2\widetilde{G}^{-1}\xi_1\widetilde{G}^{-1} - \eta_1\widetilde{G}^{-1}\xi_2\widetilde{G}^{-1},$$

$$\theta_4 = \eta_4\widetilde{G}^{-1} - \eta_3\widetilde{G}^{-1}\xi_1\widetilde{G}^{-1} - \eta_2\widetilde{G}^{-1}\xi_2\widetilde{G}^{-1} - \eta_1\widetilde{G}^{-1}\xi_3\widetilde{G}^{-1},$$

and therefore, in order to start an induction argument, the expressions to look at first are

$$\eta_2\widetilde{G}^{-1}\xi_1 + \eta_1\widetilde{G}^{-1}\xi_2 = \begin{cases} \begin{pmatrix} 0 & \begin{matrix}(P_2 - K_1\omega^{-1}K_1^t)g^{-1}K_1 \\ +K_1\omega^{-1}L_2\end{matrix} \\ L_2\omega^{-1}K_1^t & 0 \end{pmatrix} \\ \qquad \text{(even metric)} \\ \frac{1}{2}\begin{pmatrix} \begin{matrix}3K_2\kappa^{-1}P_1 + (P_1+\delta\kappa)(\kappa^{-1})^tK_2^t \\ -(P_1+\delta\kappa)(\kappa^{-1})^tL_1\kappa^{-1}P_1\end{matrix} & 0 \\ 0 & \begin{matrix}(K_2^t - L_1\kappa^{-1}P_1)(\kappa^{-1})^tL_1 \\ +L_1\kappa^{-1}K_2\end{matrix} \end{pmatrix} \\ \qquad \text{(odd metric)} \end{cases}$$

and

$$\eta_2\widetilde{G}^{-1}\xi_2 = \begin{cases} \begin{pmatrix} P_2 g^{-1}(P_2 - K_1\omega^{-1}K_1^t) & 0 \\ 0 & L_2\omega^{-1}(L_2 - K_1^t g^{-1}K_1) \end{pmatrix} \\ \qquad \text{(even metric)}, \\ \\ \frac{1}{2}\begin{pmatrix} 0 & 3K_2\kappa^{-1}(K_2 - P_1(\kappa^{-1})^t L_1) \\ K_2^t(\kappa^{-1})^t(K_2^t - L_1\kappa^{-1}P_1) & 0 \end{pmatrix} \\ \qquad \text{(odd metric)}. \end{cases}$$

It follows that from the possibilities offered by the hypotheses of Lemma 3.3, only

$$\{*\} = \begin{cases} P_2 - K_1\omega^{-1}K_1^t = 0 \text{ and } L_2 = 0 & \text{(even metric)} \\ \\ \left.\begin{array}{l} K_2 = 0 \text{ and } L_1 = 0, \text{ or} \\ K_2 = 0 \text{ and } P_1 = 0 \end{array}\right\} & \text{(odd metric)} \end{cases}$$

have the virtue of making $\theta_2 = 0$, and at the same time $\theta_3 = \eta_3\widetilde{G}^{-1}$ and $\eta_2\widetilde{G}^{-1}\xi_2\widetilde{G}^{-1} = 0$.

3.4 LEMMA: *Let $\langle\cdot,\cdot\rangle$ be a homogeneous graded metric whose Levi-Civita connection is adapted to the canonical splitting H. Assume the corresponding hypothesis from $\{*\}$ is satisfied. If for some integer $k \geq 4$, $\zeta_3 = \cdots = \zeta_{k-1} = 0$, then $\theta_2 = \theta_3 = \cdots = \theta_{k-1} = 0$, and furthermore,*

$$\theta_k = \eta_k\widetilde{G}^{-1} \qquad \textit{and} \qquad \theta_{k+1} = \eta_{k+1}\widetilde{G}^{-1} - \eta_k\widetilde{G}^{-1}\xi_1\widetilde{G}^{-1}.$$

Proof: The proof is by induction. One must also use Lemma 3.3 above and the explicit expressions for θ_2 and θ_3 to start off the induction process. ∎

THE STRUCTURE OF $\rho(\cdot)$. Using the expansion $\sum_{k\geq 1}\theta_k$ above, and the formula for ρ in terms of ∇_H obtained at the begining of this section, we may now understand the operator $D \mapsto \rho(D)$ for D ranging over the basic elements $\{\nabla_{X_a}, i_{X_a}\}$, for a given frame $\{X_a\}$. In fact, a straightforward computation shows that

$$\rho = \sum_{k\geq 1}\Big((H-\gamma)\theta_k + \sum_{i=1}^{k-1}\theta_{k-i}\theta_i\Big) = \sum_{k\geq 1}\rho_k$$

where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix of $\Gamma(D) = (-1)^{|D|}D$.

3.5 LEMMA: *Let $\langle\cdot,\cdot\rangle$ be a homogeneous graded metric whose Levi-Civita connection is adapted to the canonical splitting H. Assume the corresponding hypothesis from $\{*\}$ is satisfied. If for some integer $k \geq 4$, $\zeta_3 = \cdots = \zeta_{k-1} = 0$, then $\rho_2 = \rho_3 = \cdots = \rho_{k-1} = 0$, and furthermore, $\rho_k = (H-\gamma)\theta_k$.*

Proof: The proof is by induction. One must also use Lemmas 3.2–3.4 above and the explicit expressions for ρ_1, ρ_2 and ρ_3 to start off the induction process. ∎

We may now safely leave to the reader the details of the proof of the main proposition stated at the beginning of this section. One starts by noting that ρ_1 is identically zero for the even metric, but $\rho_1 = 0$ if and only if $L_1 = 0$ in the odd case. Using this fact one goes to the conditions under which $\rho_2 = 0$. In the even case, $\rho_2 = 0$ if and only if $L_2 = 0$, and $P_2 - K_1\omega^{-1}K_1^t = 0$, whereas in the odd case, $\rho_1 = \rho_2 = 0$ if and only if $L_1 = K_2 = 0$, etc.

4. Graded metrics of second order depth

We shall now concentrate on homogeneous graded metrics adapted to the canonical splitting H which are of second order depth. This means that there is a graded basis $\{D_i\}$ for $\operatorname{Der}\wedge\mathcal{E}$ such that,

$$\langle D_i, D_j\rangle \in \sum_{0\leq k\leq 2} \wedge^k\mathcal{E}$$

or that any of the conditions of Proposition 3.1 are satisfied. In the even case, we therefore assume that

$$\langle\cdot,\cdot\rangle = \begin{pmatrix} g+P & K \\ K^t & \omega \end{pmatrix}$$

where

$$HP = 2P, \qquad HK = K, \qquad \text{and} \qquad P = K\omega^{-1}K^t$$

and

$$\omega = -\frac{1}{2}\langle H, H\rangle \qquad \text{and} \qquad K(\chi; Y, \varphi) - K(\varphi; Y, \chi) = (\nabla_Y\omega)(\chi, \varphi).$$

In the odd case, we assume that

$$\langle\cdot,\cdot\rangle = \begin{pmatrix} P & \kappa \\ \kappa^t & 0 \end{pmatrix}$$

where the only condition is $HP = P$.

4.1 Proposition: *Let $\langle\cdot,\cdot\rangle$ be a homogeneous graded metric of second order depth which is adapted to the canonical splitting H. There exists a connection ∇' such that, with respect to the basis $\{\nabla'_X, i_\chi\}$,*

$$\langle\cdot,\cdot\rangle = \begin{cases} \begin{pmatrix} g & 0 \\ 0 & \omega \end{pmatrix} \quad \text{with} \quad \nabla'\omega = 0 & \text{(even metric)}, \\ \begin{pmatrix} 0 & \kappa \\ \kappa^t & 0 \end{pmatrix} & \text{(odd metric)}. \end{cases}$$

Proof: (1) (Even metrics) Use will be made of the isomorphisms $\omega^\sharp\colon \mathcal{E}^* \to \mathcal{E}$, and $\omega^\flat\colon \mathcal{E} \to \mathcal{E}^*$, and we shall identify an element $s \in \mathcal{E}$ with the corresponding element $s \in \operatorname{Hom}(\mathcal{E}^*, C^\infty_M)$ mapping χ into $(\chi \mid s)$. Thus

$$\omega^\sharp(\chi)(\varphi) = \omega(\chi,\varphi) \qquad \text{and} \qquad \omega\big(\omega^\flat(s),\varphi\big) = (\varphi \mid s).$$

If $\{s_\alpha\}$ is a basis for $\mathcal{E}$ and $\{\chi_\alpha\}$ is the corresponding dual basis for $\mathcal{E}^*$, then

$$\omega^\sharp(\chi) = \sum_\alpha \omega(\chi,\chi_\alpha)s_\alpha \qquad \text{and} \qquad \omega^\flat s = \sum_{\alpha,\beta}(\chi_\alpha \mid s)(\omega^{-1})_{\alpha\beta}\chi_\beta$$

where $(\omega^{-1})_{\alpha\beta}$ is the $\alpha\beta$ entry of the matrix inverse to $\omega = \big(\omega(\chi_\alpha,\chi_\beta)\big)$. In particular, the condition $P = K\omega^{-1}K^t$ translates into

$$\begin{aligned} P(\chi,\varphi;X,Y) &= K\big(\varphi;Y,\omega^\flat\big(K(\chi;X,\cdot)\big)\big) + K\big(\varphi;X,\omega^\flat\big(K(\chi;Y,\cdot)\big)\big) \\ &= -K\big(\chi;Y,\omega^\flat\big(K(\varphi;X,\cdot)\big)\big) - K\big(\chi;X,\omega^\flat\big(K(\varphi;Y,\cdot)\big)\big), \end{aligned} \tag{1}$$

equivalently into

$$P(\_,\_;X,Y) = K(\_;X,\Theta(Y)\_) = K(\_;Y,\Theta(X)\_) \tag{2}$$

where $\Theta(X) \in \operatorname{End}\mathcal{E}^*$ is given by

$$\Theta(X)\varphi = \omega^\flat\big(K(\varphi;X,\cdot)\big) \qquad \text{(even metric)}.$$

In fact, write

$$P(\_,\_;X,Y) = \sum_{\alpha,\beta} K(\_;X,\chi_\alpha)(\omega^{-1})_{\alpha\beta}K(\_;Y,\chi_\beta).$$

It is now clear that (2) is true, and (1) follows by applying $i_\chi \circ i_\varphi$ to both sides of this equation. Now use $\Theta(X)$ to change the connection ∇ by letting it be

$$\nabla'_X\varphi = \nabla_X\varphi + \Theta(X)\varphi$$

on $\mathcal{E}^*$; that is,

$$\nabla'_X = \nabla_X - i_{\Theta(X)\_}$$

as graded derivations of $\wedge\mathcal{E}$, and the result follows.

(2) (Odd metrics). For the odd metrics we make the following conventions:

$$\kappa^\sharp\colon \mathfrak{X}_M^* \to \mathcal{E}^* \qquad \text{and} \qquad \kappa^\flat\colon \mathcal{E}^* \to \mathfrak{X}_M^*$$

are defined to be inverses of each other, with

$$\kappa^\flat(\varphi)(\cdot) = \kappa(\cdot;\varphi) \qquad \text{and} \qquad \kappa\big(X;\kappa^\sharp(\eta(\cdot))\big) = \eta(X).$$

The endomorphism $\Theta(X) \in \operatorname{End}\mathcal{E}^*$ is defined by

$$\Theta(X)\varphi = \frac{1}{2}\kappa^\sharp\big(P(\varphi;X,\cdot)\big) \qquad \text{(odd metric)},$$

and the fact that $P(\_;X,Y)$ is symmetric on X and Y amounts to

$$P(\_;X,Y) = \kappa\big(Y;\Theta(X)\_\big) + \kappa\big(X;\Theta(Y)\_\big).$$

We then change the connection ∇ as before by letting $\nabla'_X\varphi = \nabla_X\varphi + \Theta(X)\varphi$, so that $\nabla'_X = \nabla_X - i_{\Theta(X)\_}$ as graded derivations of $\wedge\mathcal{E}$, and the result follows. ∎

5. Curvature of even adapted metrics of second order depth

Let ∇' be a connection on E, and let $g \in S^2(\mathfrak{X}_M^*)$ and $\omega \in \wedge^2\mathcal{E}$ be nondegenerate. We shall assume that $\langle\ \cdot\ ,\ \cdot\ \rangle$ is an even graded metric on $\wedge\mathcal{E}$ of the type just described, namely,

$$\begin{pmatrix} \langle\nabla'_X,\nabla'_Y\rangle & \langle\nabla'_X,i_\varphi\rangle \\ \langle i_\chi,\nabla'_Y\rangle & \langle i_\chi,i_\varphi\rangle \end{pmatrix} = \begin{pmatrix} g(X,Y) & 0 \\ 0 & \omega(\chi,\varphi) \end{pmatrix}, \qquad \nabla'\omega = 0.$$

Using the formula for the graded Levi–Civita connection, the formula for the Levi Civita connection of g on M, and the fact that for any $\eta \in \wedge^k\mathcal{E} \simeq \operatorname{Hom}(\wedge^k\mathcal{E}^*, C_M^\infty)$ one has $R^{\nabla'}(X,Y)\eta = -i_{R^{\nabla'}(X,Y)\_}\eta$, we obtain

$$\begin{pmatrix} \langle\boldsymbol{\nabla}_{\nabla'_X}\nabla'_Y,\nabla'_Z\rangle & \langle\boldsymbol{\nabla}_{\nabla'_X}\nabla'_Y,i_\varphi\rangle \\ \langle\boldsymbol{\nabla}_{\nabla'_X}i_\chi,\nabla'_Z\rangle & \langle\boldsymbol{\nabla}_{\nabla'_X}i_\chi,i_\varphi\rangle \end{pmatrix}$$
$$= \begin{pmatrix} g(\nabla_XY,Z) & -\frac{1}{2}\omega\big(R^{\nabla'}(X,Y)\varphi,\_\big) \\ -\frac{1}{2}\omega\big(R^{\nabla'}(Z,X)\chi,\_\big) & \omega(\nabla'_X\chi,\varphi) \end{pmatrix}.$$

Here ∇ stands for the Levi–Civita connection of g on M, and we have also used the fact that $\nabla'\omega = 0$ to conclude that $i_{R^{\nabla'}(X,Y)}\omega = 0$, and therefore

$$\omega(R^{\nabla'}(X,Y)\chi,\varphi) + \omega(\chi, R^{\nabla'}(X,Y)\varphi) = 0 \qquad \text{for all } \chi,\varphi \in \mathcal{E}^*.$$

That is, $R^{\nabla'}(X,Y) \in \mathfrak{sp}_\omega$, and we may write

$$\omega(R^{\nabla'}(X,Y)\_,\varphi) = \omega(R^{\nabla'}(X,Y)\varphi,\_), \qquad \text{etc.}$$

Similarly

$$\begin{pmatrix} \langle \mathbf{\nabla}_{i_\psi}\nabla'_Y,\nabla'_Z\rangle & \langle \mathbf{\nabla}_{i_\psi}\nabla'_Y, i_\varphi\rangle \\ \langle \mathbf{\nabla}_{i_\psi}i_\chi,\nabla'_Z\rangle & \langle \mathbf{\nabla}_{i_\psi}i_\chi, i_\varphi\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\omega(R^{\nabla'}(Y,Z)\psi,\_) & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$\mathbf{\nabla}_{\nabla'_X}\nabla'_Y = \nabla'_{\nabla_X Y} - \frac{1}{2} i_{R^{\nabla'}(X,Y)\_}, \quad \mathbf{\nabla}_{\nabla'_X} i_\varphi = -\frac{1}{2}\nabla'_{g^{-1}\left(\omega(R^{\nabla'}(\,\cdot\,,X)\varphi,\_)\right)} + i_{\nabla'_X\varphi},$$
$$\mathbf{\nabla}_{i_\chi}\nabla'_Y = -\frac{1}{2}\nabla'_{g^{-1}\left(\omega(R^{\nabla'}(\,\cdot\,,Y)\chi,\_)\right)}, \quad \mathbf{\nabla}_{i_\chi} i_\varphi = 0.$$

Remark: Note that the ∇'-component of $\mathbf{\nabla}_{\nabla'_X}\nabla'_Y$ is $\nabla_X Y$, where ∇ is the Levi-Civita connection of g. Also note that we have used a dot $\cdot$ to indicate the argument with respect to which a 1-form on M is to be transformed, via g^{-1}, into a vector field.

THE GRADED CURVATURE. It is a straightforward matter to verify from the definitions that the graded curvature tensor, when combined with the graded metric, has the following symmetries:

$$\begin{aligned}
&\langle R(D_1,D_2)D_3,D_4\rangle = -(-1)^{|D_3||D_4|}\langle R(D_1,D_2)D_4,D_3\rangle,\\
&\langle R(D_1,D_2)D_3,D_4\rangle = -(-1)^{|D_1||D_2|}\langle R(D_2,D_1)D_3,D_4\rangle,\\
&(-1)^{|D_1||D_3|}\langle R(D_1,D_2)D_3,D_4\rangle + (-1)^{|D_3||D_2|}\langle R(D_3,D_1)D_2,D_4\rangle\\
&\qquad\qquad + (-1)^{|D_2||D_1|}\langle R(D_2,D_3)D_1,D_4\rangle = 0,\\
&\langle R(D_1,D_2)D_3,D_4\rangle = (-1)^{(|D_3|+|D_4|)(|D_1|+|D_2|)}\langle R(D_3,D_4)D_1,D_2\rangle.
\end{aligned}$$

The first is true because $\mathbf{\nabla}$ is metric. The second follows directly from the definition of the graded curvature. The third is exactly Jacobi identity (using the fact that $\mathbb{V}$ is torsionless), and the fourth follows from the others. In fact, if we denote by $S(D_1, D_2, D_3, D_4)$ the left hand side of the third identity, then

$$\begin{aligned}
0 &= (-1)^{|D_1||D_3|}S(D_1, D_2, D_3, D_4) - (-1)^{(|D_1|+|D_3|)|D_4|}S(D_4, D_1, D_2, D_3) \\
&\quad + (-1)^{(|D_1|+|D_3|)|D_2|+(|D_1|+|D_2|)(|D_3|+|D_4|)}S(D_2, D_3, D_4, D_1) \\
&\quad - (-1)^{|D_1||D_3|+(|D_1|+|D_2|)(|D_3|+|D_4|)}S(D_3, D_4, D_1, D_2) \\
&= 2\big(\langle R(D_1, D_2)D_3, D_4\rangle - (-1)^{(|D_1|+|D_2|)(|D_3|+|D_4|)}\langle R(D_3, D_4)D_1, D_2\rangle\big).
\end{aligned}$$

Note that

$$\begin{aligned}
\langle R(D_1, D_2)D_3, D_4\rangle = & D_1\langle \mathbb{V}_{D_2}D_3, D_4\rangle - (-1)^{|D_1||D_2|}D_2\langle \mathbb{V}_{D_1}D_3, D_4\rangle \\
& - \langle \mathbb{V}_{[D_1,D_2]}D_3, D_4\rangle + (-1)^{|D_2||D_3|}\langle \mathbb{V}_{D_1}D_3, \mathbb{V}_{D_2}D_4\rangle \\
& - (-1)^{|D_1|(|D_2|+|D_3|)}\langle \mathbb{V}_{D_2}D_3, \mathbb{V}_{D_1}D_4\rangle.
\end{aligned}$$

It is then a straightforward matter to verify that

$$\begin{aligned}
\langle R(i_\chi, i_\varphi)i_\psi, i_\eta\rangle &= 0, \qquad \langle R(i_\chi, i_\varphi)i_\psi, \nabla'_W\rangle = 0, \\
\langle R(i_\chi, i_\varphi)\nabla'_Z, \nabla'_W\rangle &= \omega(R^{\nabla'}(Z, W)\chi, \varphi) \\
&\quad + \frac{1}{4}\omega\big(R^{\nabla'}(g^{-1}(\omega(R^{\nabla'}(\,\cdot\,, W)\varphi, _)), Z)\chi, _\big) \\
&\quad + \frac{1}{4}\omega\big(R^{\nabla'}(g^{-1}(\omega(R^{\nabla'}(\,\cdot\,, W)\chi, _)), Z)\varphi, _\big), \\
\langle R(i_\chi, \nabla'_Y)i_\varphi, \nabla'_W\rangle &= -\frac{1}{2}\omega(R^{\nabla'}(W, Y)\varphi, \chi) \\
&\quad + \frac{1}{4}\omega\big(R^{\nabla'}(g^{-1}(\omega(R^{\nabla'}(\,\cdot\,, W)\chi, _)), Y)\varphi, _\big), \\
\langle R(i_\chi, \nabla'_Y)\nabla'_Z, \nabla'_W\rangle &= \frac{1}{2}\big\{\omega\big((\nabla'_Y(R^{\nabla'}(W, Z)) \\
&\qquad - R^{\nabla'}(\nabla_Y W, Z) - R^{\nabla'}(W, \nabla_Y Z))\chi, _\big)\big\}, \\
\langle R(\nabla'_X, \nabla'_Y)\nabla'_Z, \nabla'_W\rangle &= g(R^{\nabla}(X, Y)Z, W) - \frac{1}{2}\omega(R^{\nabla'}(W, Z)R^{\nabla'}(X, Y)_, _) \\
&\quad + \frac{1}{4}\omega(R^{\nabla'}(X, Z)R^{\nabla'}(Y, W)_, _) \\
&\quad + \frac{1}{4}\omega(R^{\nabla'}(Y, Z)R^{\nabla'}(W, X)_, _).
\end{aligned}$$

Remark: One notes from these formulae that the graded manifold $(M, \wedge\mathcal{E})$ equipped with an even, adapted, second order depth metric is flat, if and only if the Riemannian base (M, g) is flat and ∇' is a flat connection on the bundle $E \to M$.

ON THE NOTION OF SECTIONAL CURVATURE. The formulae just obtained also show that any notion of sectional curvature intended to work on the $(2,2)$-dimensional planes generated by $\{\nabla'_X, \nabla'_Y, ; i_\chi, i_\varphi\}$ must give an invariant meaning to

$$\frac{\langle R(i_\chi, i_\varphi)\nabla'_Z, \nabla'_W\rangle}{F(i_\chi, i_\varphi)G(\nabla'_Z, \nabla'_W)}$$

for some Sp_ω-invariant symmetric functional F (e.g., $F(i_\chi, i_\varphi) = |\omega(\chi, \varphi)|$), and some O_g-invariant skew-symmetric functional G. In particular, since

$$\frac{\langle R(i_\chi, i_\varphi)\nabla'_Z, \nabla'_W\rangle}{F(i_\chi, i_\varphi)G(\nabla'_Z, \nabla'_W)} \equiv \frac{\omega(R^{\nabla'}(Z,W)\chi, \varphi)}{F(i_\chi, i_\varphi)G(\nabla'_Z, \nabla'_W)} \qquad \text{mod } \wedge^2 \mathcal{E}$$

and since $R^{\nabla'}(Z,W) \in \mathfrak{sp}_\omega$, an Sp_ω-invariant meaning for the left hand side immediately implies that for any $\Phi \in Sp_\omega$, $R^{\nabla'}(Z,W) \circ \Phi = \Phi \circ R^{\nabla'}(Z,W)$ and therefore $R^{\nabla'}(Z,W)$ must act like a scalar, say $\rho(Z,W)\mathbf{1}$. This, however, is too restrictive. Note, on the other hand, that the classical expression for the sectional curvature makes good sense on the graded setting for even second order depth metrics, when restricted to the $(2,0)$-dimensional planes generated by $\{\nabla'_X, \nabla'_Y\}$:

$$\kappa(X,Y) = \frac{\langle R(\nabla'_X, \nabla'_Y)\nabla'_X, \nabla'_Y\rangle}{\langle \nabla'_X, \nabla'_X\rangle\langle \nabla'_Y, \nabla'_Y\rangle - \langle \nabla'_X, \nabla'_Y\rangle^2}.$$

5.1 PROPOSITION: *Let $\langle \cdot , \cdot \rangle$ be the even graded metric adapted to the canonical splitting H corresponding to the data $\{g, \omega, \nabla'\}$. The graded manifold defined by $\wedge\mathcal{E}$ has constant graded sectional curvature on the $(2,0)$-dimensional planes generated by $\{\nabla'_X, \nabla'_Y\}$, if and only if (M,g) is a Riemannian manifold of constant curvature, and the curvature of ∇' in E satisfies $R^{\nabla'}(X,Y)R^{\nabla'}(X,Y) = 0$, for all X and Y.*

Proof: This follows immediately from the formulae above. The first assertion is clear. But then, the 2-form

$$\frac{3}{4(g(X,X)g(Y,Y) - g(X,Y)^2)}\omega(R^{\nabla'}(X,Y)R^{\nabla'}(X,Y)\_, \_)$$

must be constant; hence, zero. Thus $R^{\nabla'}(X,Y)R^{\nabla'}(X,Y) = 0$ for all X and Y.

∎

THE GRADED RICCI TENSOR. The graded Ricci tensor GRic is defined as the graded tensor whose value on the pair of graded derivations (D_1, D_2) is given by the supertrace of the endomorphism

$$D \mapsto R(D, D_1)D_2.$$

The supertrace of this endomorphism may be computed with the aid of the graded metric in terms of a given basis. In fact, if we write

$$R(\nabla'_{X_a}, D_1)D_2 = \sum_{c=1}^{m} A_{ac}\nabla'_{X_c} + \sum_{\beta=1}^{n} B_{a\beta} i_{X_\beta},$$
$$R(i_{X_\alpha}, D_1)D_2 = \sum_{c=1}^{m} C_{\alpha c}\nabla'_{X_c} + \sum_{\beta=1}^{n} D_{\alpha\beta} i_{X_\beta},$$

then

$$\begin{pmatrix} \langle R(\nabla'_{X_a}, D_1)D_2, \nabla'_{X_b}\rangle & \langle R(\nabla'_{X_a}, D_1)D_2, i_{X_\beta}\rangle \\ \langle R(i_{X_\alpha}, D_1)D_2, \nabla'_{X_b}\rangle & \langle R(i_{X_\alpha}, D_1)D_2, i_{X_\beta}\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \langle \nabla'_{X_c}, \nabla'_{X_b}\rangle & \langle \nabla'_{X_c}, i_{X_\beta}\rangle \\ \langle i_{X_\gamma}, \nabla'_{X_b}\rangle & \langle i_{X_\gamma}, i_{X_\beta}\rangle \end{pmatrix}$$

and therefore

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \langle R(\nabla'_{\cdot}, D_1)D_2, \nabla'_{\cdot}\rangle & \langle R(\nabla'_{\cdot}, D_1)D_2, i_{\cdot}\rangle \\ \langle R(i_{\cdot}, D_1)D_2, \nabla'_{\cdot}\rangle & \langle R(i_{\cdot}, D_1)D_2, i_{\cdot}\rangle \end{pmatrix}\langle\,\cdot\,,\,\cdot\,\rangle^{-1}.$$

If the basis is orthonormal, then the computation of the inverse is easier and the computation of the graded Ricci tensor simplifies considerably. But when no orthonormal basis is given *a priori*, the graded Ricci is computed from

$$\mathrm{GRic}(D_1, D_2) = \mathrm{Str}\left\{\begin{pmatrix} \langle R(\nabla'_{\cdot}, D_1)D_2, \nabla'_{\cdot}\rangle & \langle R(\nabla'_{\cdot}, D_1)D_2, i_{\cdot}\rangle \\ \langle R(i_{\cdot}, D_1)D_2, \nabla'_{\cdot}\rangle & \langle R(i_{\cdot}, D_1)D_2, i_{\cdot}\rangle \end{pmatrix}\langle\,\cdot\,,\,\cdot\,\rangle^{-1}\right\}.$$

The proof that $\mathrm{GRic}(D_1, D_2) = (-1)^{|D_1||D_2|}\,\mathrm{GRic}(D_2, D_1)$ boils down to

$$(-1)^{|D_1|+|D_2|}\sum_{\alpha,\beta}\langle R(D_1, D_2)i_{X_\alpha}, i_{X_\beta}\rangle(\omega^{-1})_{\alpha\beta} - \sum_{a,b}\langle R(D_1, D_2)\nabla'_{X_a}, \nabla'_{X_b}\rangle(g^{-1})_{ab} = 0$$

which is obviously satisfied in the even case. For the odd metrics the graded symmetry of the Ricci tensor amounts to verifying the equality

$$\left(1+(-1)^{|D_1|+|D_2|}\right)\sum_{a,\beta}\langle R(D_1,D_2)\nabla'_{X_a}, i_{\chi_\beta}\rangle(\kappa^\flat)_{\beta a}=0,$$

which is not so evident and has to be verified directly after computing the matrix coefficients $\langle R(D_1,D_2)\nabla'_{X_a}, i_{\chi_\beta}\rangle$, and taking the contraction with $(\kappa^\flat)_{\beta a}$. In our case, for the simplest even adapted metrics, a direct computation yields

$$\begin{aligned}
\mathrm{GRic}(i_\chi, i_\varphi) &= -\frac{1}{4}\omega\big(R^{\nabla'}(g^{-1}(\omega(R^{\nabla'}(\,\cdot\,,X_b)\chi,_)),X_a)\varphi,_\big)(g^{-1})_{ba},\\
\mathrm{GRic}(i_\chi,\nabla'_Y) &= \frac{1}{2}\omega\big(\{\nabla'_{X_a}(R^{\nabla'}(Y,X_b))-R^{\nabla'}(\nabla_{X_a}Y,X_b)\\
&\qquad -R^{\nabla'}(Y,\nabla_{X_a}X_b)\}\chi,_\big)(g^{-1})_{ba},\\
\mathrm{GRic}(\nabla'_X,\nabla'_Y) &= \mathrm{Ric}^{\nabla}(X,Y)+\frac{3}{4}\omega(R^{\nabla'}(Y,X_a)(g^{-1})_{ab}R^{\nabla'}(X_b,X)_,_)\\
&\quad +\frac{1}{2}\omega\big(R^{\nabla'}(g^{-1}(\omega(R^{\nabla'}(\,\cdot\,,Y)\chi_\alpha,_)),X)\chi_\beta,_\big)(\omega^{-1})_{\beta\alpha},
\end{aligned}$$

where $\mathrm{Ric}^{\nabla}(X,Y)$ is the Ricci tensor of the Levi-Civita connection ∇ of g, and there is a sum understood over repeated indices.

5.2 PROPOSITION: *Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the even graded metric adapted to the canonical splitting H corresponding to the data $\{g,\omega,\nabla'\}$. Let ∇ be the Levi-Civita connection of g. If the graded manifold defined by $\wedge\mathcal{E}$ is Einstein, then* $\mathrm{Ric}^{\nabla}(X,Y)=0$.

Proof: For the graded manifold $\wedge\mathcal{E}$ to be Einstein, a $\lambda=\lambda_0+\lambda_2+\cdots\in\wedge\mathcal{E}$ is needed such that $\lambda\langle D_1,D_2\rangle=\mathrm{GRic}(D_1,D_2)$. Taking $(D_1,D_2)=(i_\chi,i_\varphi)$, the formulae above shows that $\lambda_0=0$. But then, this implies that $\mathrm{Ric}^{\nabla}(X,Y)=\lambda_0 g(X,Y)=0$ when $(D_1,D_2)=(\nabla'_X,\nabla'_Y)$. ∎

5.3 PROPOSITION: *Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the even graded metric adapted to the canonical splitting H corresponding to the data $\{g,\omega,\nabla'\}$. Let ∇ be the Levi-Civita connection of g. The graded manifold defined by $\wedge\mathcal{E}$ is Ricci flat in the graded sense if and only if the Riemannian base manifold is Ricci flat ($\mathrm{Ric}^{\nabla}(X,Y)=0$), and the following two equations are satisfied:*

$$\sum_{a,b}(g^{-1})_{ba}\{\nabla'_{X_a}(R^{\nabla'}(Y,X_b))-R^{\nabla'}(\nabla_{X_a}Y,X_b)-R^{\nabla'}(Y,\nabla_{X_a}X_b)\}=0$$

and

$$\sum_{a,b}(g^{-1})_{ba}R^{\nabla'}(X,X_a)R^{\nabla'}(Y,X_b)=0.$$

Proof: Again, the assertion is a straightforward consequence of the explicit formulae for the graded Ricci tensor. Clearly $\mathrm{Ric}^{\nabla}(X,Y)$ has to vanish independently. Now, the first equation in the statement is just the vanishing of $\mathrm{GRic}(i_X,\nabla'_Y)$. The second equation is equivalent to the vanishing of the second term in the right of $\mathrm{GRic}(\nabla'_X,\nabla'_Y)$, and this in turn implies the vanishing of $\mathrm{GRic}(i_X,i_\varphi)$, which is the same as the vanishing of the third term in the right of $\mathrm{GRic}(\nabla'_X,\nabla'_Y)$. ∎

6. Adapted metrics on differential forms

We now want to specialize some of our results to the graded manifold defined by the algebra $\Omega(M)$ of differential forms on the smooth manifold M and to closely investigate the role played by the exterior derivative d in the presence of adapted graded metrics. Our first result is the following:

6.1 PROPOSITION: *Let $\mathfrak{g}$ be the $(1,1)$-dimensional Lie superalgebra generated by the adapted derivation $H=i_{\mathrm{Id}}$, and the exterior derivative* d*. Then, $\mathfrak{g}$ does not exponentiate to isometries for any graded metric on $\Omega(M)$. Specifically, $H=i_{\mathrm{Id}}$ cannot generate isometries, nor conformal transformations for any graded metric adapted to it.*

Remark: On the other hand, it has been shown in [9] that d itself does generate isometries for a class of odd graded metrics.

Proof: We shall prove that H cannot generate conformal transformations. Let ∇ be any linear connection in TM. Let $\lambda\in\Omega(M)$ be an invertible **even** differential form, and suppose that for any $X,Y\in\mathfrak{X}(M)$,

$$(H\lambda)\langle i_X,i_Y\rangle+\lambda H\langle i_X,i_Y\rangle=\langle[H,i_X],i_Y\rangle+\langle i_X,[H,i_Y]\rangle,$$
$$(H\lambda)\langle\nabla_X,i_Y\rangle+\lambda H\langle\nabla_X,i_Y\rangle=\langle[H,\nabla_X],i_Y\rangle+\langle\nabla_X,[H,i_Y]\rangle,$$
$$(H\lambda)\langle\nabla_X,\nabla_Y\rangle+\lambda H\langle\nabla_X,\nabla_Y\rangle=\langle[H,\nabla_X],\nabla_Y\rangle+\langle\nabla_X,[H,\nabla_Y]\rangle.$$

The first of these equations implies

$$\lambda H\langle i_X,i_Y\rangle=-(2+H\lambda)\langle i_X,i_Y\rangle,$$

where we have used the fact that i_X and i_Y have degree -1 for the $\mathbf{Z}$-grading generated by H. In particular, $-(2+H\lambda)/\lambda$ is an eigenvalue of H; that is, $H\lambda = -(2+k\lambda)$, for some nonnegative integer k. On the other hand, the second condition for H to generate (local) conformal transformations implies

$$\lambda H\langle \nabla_X, i_Y\rangle = -(1+H\lambda)\langle \nabla_X, i_Y\rangle$$

and this time $H\lambda = -(1+k'\lambda)$ (using the fact that ∇_X has degree zero). Hence, $(k-k')\lambda = -1$; that is, λ is a constant. In particular, $H\lambda = 0$ and therefore,

$$\lambda H\langle i_X, i_Y\rangle = -2\langle i_X, i_Y\rangle \qquad \text{and} \qquad \lambda H\langle \nabla_X, i_Y\rangle = -\langle \nabla_X, i_Y\rangle,$$

which imply $\langle i_X, i_Y\rangle = 0$ and $\langle \nabla_X, i_Y\rangle = 0$, but these two conditions contradict the fact that the graded metric $\langle\,\cdot\,,\,\cdot\,\rangle$ is nondegenerate. ∎

Remark: For the sake of completeness (and uniformity in our use of ∇) we shall now investigate whether or not d generates (local) conformal transformations. We shall need the results contained in the next Lemma based on the decomposition of graded derivations of $\Omega(M)$ as described in [4] and [5].

6.2 LEMMA:

(1) *Let ∇ be a torsionless connection in TM. Then*

$$[\nabla_X, \mathrm{d}] = -\nabla_{\nabla X} + i_{R(\_,\_)X} \qquad \text{and} \qquad [\mathrm{d}, i_X] = \nabla_X + i_{\nabla X}$$

as graded derivations of $\Omega(M)$.

(2) *Let ∇' be an afine connection on TM. Let $K \in \Omega^1(M;TM)$ and $L \in \Omega^2(M;TM)$ be the differential forms with values on the tangent bundle of M defined by*

$$K(X) = X \qquad \text{and} \qquad L(X,Y) = T^{\nabla'}(X,Y) = \nabla'_X Y - \nabla'_Y X - [X,Y].$$

Then, $\mathrm{d} = \nabla'_{K(\_)} + i_{L(\_,\_)}$.

Proof: (1) Let $f \in \Omega^0(M) = C^\infty(M)$. Then, for any $Z \in \mathfrak{X}(M)$,

$$\begin{aligned}([\nabla_X, \mathrm{d}]f)(Z) &= \big(\nabla_X(\mathrm{d}f)\big)(Z) - \big(\mathrm{d}(\nabla_X f)\big)(Z)\\ &= X\big((df)(Z)\big) - (\mathrm{d}f)(\nabla_X Z) - Z(Xf)\\ &= X(Zf) - Z(Xf) - (\nabla_X Z)f = ([X,Z] - \nabla_X Z)f = -(\nabla_Z X)f.\end{aligned}$$

Therefore, $[\nabla_X, \mathrm{d}] = -\nabla_{\nabla X} + i_{K(\_,\_;X)}$ for some 2-form $K(\_,\_;X)$ with values in $\mathfrak{X}(M)$. Now let $\theta \in \Omega^1(M)$. Then, for any $Z, W \in \mathfrak{X}(M)$,

$$([\nabla_X, \mathrm{d}]\theta)(Z, W) = \nabla_W X(\theta(Z)) - \nabla_Z X(\theta(W)).$$

On the other hand, it is easy to check that

$$(\nabla_{\nabla X}\theta)(Z, W) = \nabla_Z X(\theta(W)) - \nabla_W X(\theta(Z)) - \theta(\nabla_{\nabla_Z X} W - \nabla_{\nabla_W X} Z).$$

Thus

$$-(\nabla_{\nabla X}\theta)(Z, W) - \theta(\nabla_{\nabla_Z X} W - \nabla_{\nabla_W X} Z) = ([\nabla_X, \mathrm{d}]\theta)(Z, W)$$

and therefore

$$(i_{K(\_,\_;X)}\theta)(Z, W) = -\theta(\nabla_{\nabla_Z X} W - \nabla_{\nabla_W X} Z).$$

But now

$$\begin{aligned}\nabla_{\nabla_Z X} W - \nabla_{\nabla_W X} Z &= \nabla_W \nabla_Z X - \nabla_Z \nabla_W X + [\nabla_Z X, W] - [\nabla_W X, Z] \\ &= R(W, Z)X + \nabla_{[W,Z]} X + [\nabla_Z X, W] - [\nabla_W X, Z].\end{aligned}$$

Finally note that, after some easy computations,

$$\nabla_{[W,Z]} X + [\nabla_Z X, W] - [\nabla_W X, Z] = 0.$$

Therefore

$$\begin{aligned}(i_{K(\_,\_;X)}\theta)(Z, W) &= -\theta(\nabla_{\nabla_Z X} W - \nabla_{\nabla_W X} Z) \\ &= -\theta(R(W, Z)X) = i_{R(Z,W)X}\theta = (i_{R(\_,\_)X}\theta)(Z, W)\end{aligned}$$

from which the first formula in the statement follows. The second is proved in a similar manner.

(2) Let $f \in \Omega^0(M)$. Note that $\left(\nabla'_{K(\_)} + i_{L(\_,\_)}\right)f \in \Omega^1(M)$, and since $i_{L(\_,\_)}f = 0$ we have, on the one hand,

$$\left(\left(\nabla'_{K(\_)} + i_{L(\_,\_)}\right)f\right)(X) = (\nabla'_{K(\_)}f)(X) = \nabla'_{K(X)}f = K(X)f.$$

But, on the other hand, we know that $(\mathrm{d}f)(X) = Xf$, so that $K(X) = X$ as claimed.

Now let $\theta \in \Omega^1(M)$. Note that $(\nabla'_{K(\_)} + i_{L(\_,\_)})\theta \in \Omega^2(M)$, and

$$\begin{aligned}\big((\nabla'_{K(\_)} + i_{L(\_,\_)})\theta\big)(X,Y) =&(\nabla'_X\theta)(Y) - (\nabla'_Y\theta)(X) + \theta\big(L(X,Y)\big)\\ =&X\big(\theta(Y)\big) - Y\big(\theta(X)\big)\\ &- \theta\big(\nabla'_X Y - \nabla'_Y X + L(X,Y)\big)\end{aligned}$$

But $\nabla'_X Y - \nabla'_Y X = [X,Y] + T^{\nabla'}(X,Y)$. Therefore

$$\big((\nabla'_{K(\_)} + i_{L(\_,\_)})\theta\big)(X,Y) = \mathrm{d}\,\theta(X,Y) + \theta\big(L(X,Y) - T^{\nabla'}(X,Y)\big)$$

and therefore $L(X,Y) = T^{\nabla'}(X,Y)$. ∎

Let $\lambda \in \Omega(M)$ be an **even** invertible differential form, and suppose that for any $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned}(\mathrm{d}\,\lambda)\langle i_X, i_Y\rangle + \lambda\,\mathrm{d}\langle i_X, i_Y\rangle &= \langle[\mathrm{d}, i_X], i_Y\rangle - \langle i_X, [\mathrm{d}, i_Y]\rangle,\\ (\mathrm{d}\,\lambda)\langle \nabla_X, i_Y\rangle + \lambda\,\mathrm{d}\langle \nabla_X, i_Y\rangle &= \langle[\mathrm{d}, \nabla_X], i_Y\rangle + \langle \nabla_X, [\mathrm{d}, i_Y]\rangle,\\ (\mathrm{d}\,\lambda)\langle \nabla_X, \nabla_Y\rangle + \lambda\,\mathrm{d}\langle \nabla_X, \nabla_Y\rangle &= \langle[\mathrm{d}, \nabla_X], \nabla_Y\rangle + \langle \nabla_X, [\mathrm{d}, \nabla_Y]\rangle.\end{aligned}$$

Then, the first of the conditions for d to generate local conformal transformations yields

$$(\mathrm{d}\,\lambda)\langle i_X, i_Y\rangle + \lambda\,\mathrm{d}\langle i_X, i_Y\rangle = \langle \nabla_X, i_Y\rangle - \langle i_X, \nabla_Y\rangle + \langle i_{\nabla X}, i_Y\rangle - \langle i_X, i_{\nabla Y}\rangle.$$

The second of the conditions yields

$$(\mathrm{d}\,\lambda)\langle \nabla_X, i_Y\rangle + \lambda\,\mathrm{d}\langle \nabla_X, i_Y\rangle = \langle \nabla_X, \nabla_Y\rangle - \langle i_{R(\_,\_)X}, i_Y\rangle + \langle \nabla_{\nabla X}, i_Y\rangle + \langle \nabla_X, i_{\nabla Y}\rangle$$

and finally, we obtain from the third,

$$\begin{aligned}(\mathrm{d}\,\lambda)\langle \nabla_X, \nabla_Y\rangle + \lambda\,\mathrm{d}\langle \nabla_X, \nabla_Y\rangle =&\langle \nabla_{\nabla X}, \nabla_Y\rangle + \langle \nabla_X, \nabla_{\nabla Y}\rangle\\ &- \langle i_{R(\_,\_)X}, \nabla_Y\rangle - \langle \nabla_X, i_{R(\_,\_)Y}\rangle.\end{aligned}$$

Note that there is no $\Omega^0(M)$ component in $(\mathrm{d}\,\lambda)\langle i_X, i_Y\rangle + \lambda\,\mathrm{d}\langle i_X, i_Y\rangle$. Therefore, the corresponding $\Omega^0(M)$ component appearing in the right hand side of it must vanish identically. That is, $K_0(X,Y) = K_0(Y,X)$. The same argument works for $(\mathrm{d}\,\lambda)\langle \nabla_X, i_Y\rangle + \lambda\,\mathrm{d}\langle \nabla_X, i_Y\rangle$. This expression has no $\Omega^0(M)$ component; hence, $P_0(X,Y) = 0$. In particular, the graded metric cannot be even, and if it is odd (i.e., homogeneous) then K_0 must be nondegenerate and therefore defines a Riemannian metric on M, as found in [9] by other means.

REPRESENTATION OF $\mathfrak{g}$ BY $\mathbf{\nabla}$. We shall restrict ourselves for the moment to adapted metrics of second order depth in order to show that, even in this simpler case, the Lie algebra $\mathfrak{g}$ cannot in general be represented by $\mathbf{\nabla}$. We shall look at the curvature coefficients $\langle R(H,\mathrm{d})D_1, D_2\rangle$ and $\langle R(\mathrm{d},\mathrm{d})D_1, D_2\rangle$ for D_1 and D_2 ranging over the basic set of derivations $\{\nabla'_X, i_Y\}$, and with ∇' be chosen so that

$$\nabla'_X Y = \nabla_X Y + \Theta(X)Y$$

with ∇ being the Levi–Civita connection of a metric on the base ($g = P_0$ or $\kappa = K_0$, depending on whether the graded metric is even or odd, respectively). so that $T^{\nabla'}(X,Y) = \Theta(X)Y - \Theta(Y)X$. Note first that

$$\begin{aligned}\langle R(H,\mathrm{d})D_1, D_2\rangle = H\langle \mathbf{\nabla}_{\mathrm{d}} D_1, D_2\rangle - \langle \mathbf{\nabla}_{\mathrm{d}} D_1, \mathbf{\nabla}_H D_2\rangle - \langle \mathbf{\nabla}_{\mathrm{d}} D_1, D_2\rangle \\ - \mathrm{d}\langle \mathbf{\nabla}_H D_1, D_2\rangle + (-1)^{|D_1|}\langle \mathbf{\nabla}_H D_1, \mathbf{\nabla}_{\mathrm{d}} D_2\rangle.\end{aligned}$$

But we have already found that second order depth adapted metrics have the property that $\mathbf{\nabla}_H D = 0$ for $D \in \{\nabla'_X, i_Y\}$. Therefore

$$\langle R(H,\mathrm{d})D_1, D_2\rangle = H\langle \mathbf{\nabla}_{\mathrm{d}} D_1, D_2\rangle - \langle \mathbf{\nabla}_{\mathrm{d}} D_1, D_2\rangle, \qquad D_i \in \{\nabla_X, i_Y\}.$$

In particular, $\langle R(H,\mathrm{d})D_1, D_2\rangle$ will be non-zero, as soon as the coefficient $\langle \mathbf{\nabla}_{\mathrm{d}} D_1, D_2\rangle$ contains a non-zero 2-form. To investigate this we may write d as in Lemma 6.2 (2), and use the Christoffel symbols at the beginning of §5 for $E = T^*M$, together with the definition of the graded metric. Thus, for example, we find

$$\langle \mathbf{\nabla}_{\mathrm{d}} \nabla'_X, i_Y\rangle = -\frac{1}{2}\langle i_{R^{\nabla'}(K(\_),X)\_}, i_Y\rangle = -\frac{1}{2}\omega\big(R^{\nabla'}(K(\_),X)\_, Y\big).$$

But it is now evident that $\langle R(H,\mathrm{d})\nabla'_X, i_Y\rangle$ will not be zero, since this 2-form is in general non-zero. The reader may verify, for example, that even though $\langle \mathbf{\nabla}_{\mathrm{d}} \nabla'_X, \nabla'_Y\rangle$ would make the 3-form $\omega(R^{\nabla'}(X,Y)L(\_,\_),\_)$ appear, it is nevertheless zero (as a consequence of $\nabla'\omega = 0$). Similarly, an easier computation shows that $\langle \mathbf{\nabla}_{\mathrm{d}} i_X, i_Y\rangle$ is always a 1-form, but $\langle \mathbf{\nabla}_{\mathrm{d}} i_X, \nabla'_Y\rangle$ is again a scalar multiple of the 2-form $\omega\big(R^{\nabla'}(K(\_),X)\_, Y\big)$. On the other hand, we have

$$\langle R(\mathrm{d}.\,\mathrm{d})D_1, D_2\rangle = 2(\mathrm{d}\langle \mathbf{\nabla}_{\mathrm{d}} D_1, D_2\rangle + (-1)^{|D_1|}\langle \mathbf{\nabla}_{\mathrm{d}} D_1, \mathbf{\nabla}_{\mathrm{d}} D_2\rangle),$$

which again may be explicitly computed from the Christoffel symbols and the definition of the graded metric. More tedious and lengthier computations show that these are in general non-zero because $\mathrm{d}\,\omega$ is not necessarily zero, and because the same non-zero 2-form above appears.

ON THE STRUCTURE OF $\mathbb{V}_{\mathrm{d}}$. From the definition of the graded Levi–Civita connection, one may compute the connection coefficients $\langle \mathbb{V}_{\mathrm{d}}\nabla_X, \nabla_Y\rangle$, $\langle \mathbb{V}_{\mathrm{d}}\nabla_X, i_Y\rangle$, $\langle \mathbb{V}_{\mathrm{d}} i_X, \nabla_Y\rangle$, and $\langle \mathbb{V}_{\mathrm{d}} i_X, i_Y\rangle$. The results, however, are rather complicated and not particularly useful (as they were for $\mathbb{V}_H$ in §3). Nevertheless, there are some general conclusions that can be drawn for d and $\mathbb{V}_{\mathrm{d}}$: For example, $\langle \mathrm{d}, \mathrm{d}\rangle = 0$, which follows easily by writing $\mathrm{d} = \nabla_K$ as in Lemma 6.2 above with a torsionless connection ∇.

6.3 LEMA: *Let $\langle\ \cdot\ ,\ \cdot\ \rangle$ be any homogeneous graded metric on $\Omega(M)$, and let $\mathbb{V}$ be its Levi–Civita connection. Then, $\mathbb{V}_{\mathrm{d}}\,\mathrm{d} = 0$.*

Proof: This is a straighforward calculation from the formula for the Levi–Civita connection: For any homogeneous derivation of degree $|D|$, we have, $2\langle \mathbb{V}_{\mathrm{d}}\,\mathrm{d}, D\rangle = -D\langle \mathrm{d}, \mathrm{d}\rangle$ and the assertion follows from the fact that for any graded metric $\langle \mathrm{d}, \mathrm{d}\rangle = 0$. ∎

6.4 LEMA: *Let $\langle\ \cdot\ ,\ \cdot\ \rangle$ be any homogeneous graded metric on $\Omega(M)$, and let $\mathbb{V}$ be its Levi–Civita connection. Let*

$$\rho(\mathrm{d}, D) = \mathrm{d}\langle H, D\rangle - (-1)^{|D|} D\langle H, \mathrm{d}\rangle - \langle [\mathrm{d}, D], H\rangle.$$

Then,

$$\begin{aligned} 2\langle \mathbb{V}_H\,\mathrm{d}, D\rangle &= H\langle \mathrm{d}, D\rangle + (1 - |D|)\langle \mathrm{d}, D\rangle + \rho(\mathrm{d}, D), \\ 2\langle \mathbb{V}_{\mathrm{d}} H, D\rangle &= H\langle \mathrm{d}, D\rangle - (1 + |D|)\langle \mathrm{d}, D\rangle + \rho(\mathrm{d}, D). \end{aligned}$$

Moreover, $\rho(\mathrm{d}, D) = 0$ identically when $\langle\ \cdot\ ,\ \cdot\ \rangle$ is H-adapted and even. When the graded metric is H-adapted and odd, we have

$$\begin{aligned} \rho(\mathrm{d}, i_X) &= -K_0(\_; X), \\ \rho(\mathrm{d}, \nabla_X) &= \mathrm{d}\big(K_0(X; \_)\big) - (\nabla_X K_0{}^a)(\_; \_) - K_0(\nabla X; \_), \end{aligned}$$

where $K_0{}^a(Z; W) = K_0(Z, W) - K_0(W, Z)$. In particular, if K_0 is a Riemannian metric and ∇ its Levi–Civita connection, then $\rho(\mathrm{d}, \nabla_X) = 0$.

Proof: Using the formula for the Levi–Civita connection we have

$$\begin{aligned} 2\langle \mathbb{V}_H\,\mathrm{d}, D\rangle = {} & H\langle \mathrm{d}, D\rangle + (1 - |D|)\langle \mathrm{d}, D\rangle \\ & + \mathrm{d}\langle H, D\rangle - (-1)^{|D|} D\langle H, \mathrm{d}\rangle - \langle [\mathrm{d}, D], H\rangle, \\ 2\langle \mathbb{V}_{\mathrm{d}} H, D\rangle = {} & H\langle \mathrm{d}, D\rangle - (1 + |D|)\langle \mathrm{d}, D\rangle \\ & + \mathrm{d}\langle H, D\rangle - (-1)^{|D|} D\langle H, \mathrm{d}\rangle - \langle H, [\mathrm{d}, D]\rangle, \end{aligned}$$

thus proving the formulae of the first part of the statement. As for the evaluation of $\rho(\mathrm{d}, D)$, the assertion for even adapted metrics follows from the fact that $2\langle H, D\rangle = D\langle H, H\rangle$. For odd adapted metrics, we know that $\langle H, i_X\rangle = 0$. This implies $\langle H, i_K\rangle = 0$, for any differential form K with values in TM. On the other hand, we know that $\langle H, \nabla_X\rangle \in \Omega^1(M)$. In fact,$\langle H, \nabla_X\rangle = K_0(X; \_)$. It is then easy to check—e.g., writing $\mathrm{d} = \theta^a \nabla_{X_a}$ (summation convention)—that

$$(i_X\langle H, \mathrm{d}\rangle)(W) = K_0(X; W) - K_0(W, X),$$

and since $[\mathrm{d}, i_X] = \nabla_X + i_{\nabla X}$, it follows that

$$\rho(\mathrm{d}, i_X) = \mathrm{d}\langle H, i_X\rangle + i_X\langle H, \mathrm{d}\rangle - \langle[\mathrm{d}, i_X], H\rangle = -K_0(\_; X).$$

Similarly, a straightforward computation yields the formula for $\rho(\mathrm{d}, \nabla_X)$. ∎

6.5 Lemma: *Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a homogeneous, H-adapted, graded metric on $\Omega(M)$. Then*

$$\langle \mathrm{d}, \nabla_X\rangle = \begin{cases} P_0(\_, X) + P_{e+}(\_; \_, X) & \textit{(even metric)}, \\ -P_1{}^a(\_; \_, X) + P_{o+}(\_; \_, X) & \textit{(odd metric)}, \end{cases}$$

where P_{e+} and P_{o+} belong to the Kernel of some Young symmetrizers (analogous to those used in Proposition 2.3). Similarly,

$$\langle \mathrm{d}, i_X\rangle = \begin{cases} -K_1{}^a(\_; \_, X) + K_{o+}(\_; \_, X) & \textit{(even metric)}, \\ K_0(\_; X) + K_{e+}(\_; \_, X) & \textit{(odd metric)}. \end{cases}$$

Adapted metrics on the plane H-d. We finally want to look at the restriction of an adapted metric to the $\Omega(M)$-span of the derivations H and d. Since $\langle \mathrm{d}, \mathrm{d}\rangle = 0$, we may write

$$\begin{pmatrix} \langle H, H\rangle & \langle H, \mathrm{d}\rangle \\ \langle \mathrm{d}, H\rangle & \langle \mathrm{d}, \mathrm{d}\rangle \end{pmatrix} = \begin{pmatrix} \omega & \eta \\ \eta & 0 \end{pmatrix}.$$

Now, we would like to define a degree $+1$ derivation d' in the $\mathbb{Z}$-grading generated by H, and in the $\Omega(M)$-span of H and d, such that $\langle H, \mathrm{d}'\rangle = 0$, with $\langle \mathrm{d}', \mathrm{d}'\rangle = 0$ and $\mathrm{d}' \circ \mathrm{d}' = 0$.

6.6 Lemma: *Let $\mathrm{d}' = f\,\mathrm{d} + \beta H$. Then*

$$[H, \mathrm{d}'] = \mathrm{d}' \iff f \in \Omega^0(M) \quad \textit{and} \quad \beta \in \Omega^1(M).$$

In that case, $\langle \mathrm{d}', \mathrm{d}' \rangle = 0$ *and* $\mathrm{d}' \circ \mathrm{d}' = 0 \iff \beta = -\mathrm{d} f$.

Proof: This is a simple computation: First, note that

$$[H, \mathrm{d}'] = \big(f + (Hf)\big)\,\mathrm{d} + (H\beta)H,$$

and the latter is equal to $f\,\mathrm{d} + \beta H$ if and only if $Hf = 0$ and $H\beta = \beta$, and the first statement follows. For the second, note that

$$(f\,\mathrm{d} + \beta H) \circ (f\,\mathrm{d} + \beta H) = f(\mathrm{d} f + \beta)\,\mathrm{d} + \mathrm{d}\,\beta H. \quad \blacksquare$$

Remark: The question arises as to what is the relationship between the cohomology of the complex $\{\Omega(M); \mathrm{d}\}$ and that of $\{\Omega(M); \mathrm{d}'\}$. The answer is given in the following Lemma (for related work on **shifted cohomology** see also [14]).

6.7 LEMMA: *Let $f \in \Omega^0(M)$ be a nowhere vanishing function on M. Let $H^k(M)$ be the k-th cohomology group of the de Rham complex, and let $H^k_f(M)$ be the k-th cohomology group of the complex $\{\Omega(M); \mathrm{d}'\}$, when $\mathrm{d}' = f\,\mathrm{d} - (\mathrm{d} f)H$. Then*

$$[\eta] \in H^k_f(M) \iff \left[\frac{1}{f^k}\eta\right] \in H^k(M).$$

Proof: It is a simple computation. $\blacksquare$

6.8 PROPOSITION: *Suppose $\mathrm{d}' = f\,\mathrm{d} - (\mathrm{d} f)H$ for some non-vanishing function f on M.*

(1) *Let $\langle \cdot\,, \cdot \rangle$ be an* **even** *adapted metric. Then*

$$\langle \mathrm{d}', H \rangle = 0 \iff \mathrm{d}\,\omega = 2\frac{\mathrm{d} f}{f}\omega \iff \left[\frac{1}{f^2}\omega\right] \in H^2(M).$$

(2) *Let $\langle \cdot\,, \cdot \rangle$ be an* **odd** *adapted metric. Then*

$$\langle \mathrm{d}', H \rangle = 0 \iff \langle \mathrm{d}, H \rangle = 0 \iff K_0(Z, W) = K_0(W, Z),$$

that is, K_0 defines a Riemannian metric if and only if the H-d plane is isotropic.

Proof: If the metric is adapted and even, then $\langle H, H \rangle = \omega$ implies $\langle H, \mathrm{d}' \rangle = \frac{1}{2}\mathrm{d}'\,\omega$. Since $H\omega = 2\omega$, we have

$$\frac{1}{2}\mathrm{d}'\,\omega = \frac{1}{2}(f\,\mathrm{d}\,\omega - 2(\mathrm{d} f)\omega).$$

When the adapted metric is odd, we know that $|D| = -1$ implies $\langle H, D\rangle = 0$. In particular, since $H = i_{\mathrm{Id}} = \theta^a i_{X_a}$ (summation convention), we clearly have $\langle H, H\rangle = 0$. On the other hand, we also know that $|D| = 0$ implies $\langle H, D\rangle \in \Omega^1(M)$, and we may write

$$A(X, \_) = \langle H, \nabla_X\rangle.$$

Writing $H = \theta^a i_{X_a}$ (summation convention), it is easily seen that

$$A(X, \_) = \langle H, \nabla_X\rangle = K_0(X, \_).$$

On the other hand, writing $\mathrm{d} = \theta^a \nabla_{X_a}$ (summation convention) we have

$$\eta(Z, W) = \langle H, \mathrm{d}\rangle(Z, W) = K_0(Z, W) - K_0(W, Z).$$

Finally, if $\mathrm{d}' = f\,\mathrm{d} - (\mathrm{d} f)H$, we obtain $\langle H, \mathrm{d}'\rangle = f\eta$ and the second statement follows. ∎

Remark: Note what are the conditions on the simplest adapted even graded metrics to achieve this situation. If $\{X_a\}$ is a local frame in M, and $\{\theta^a\}$ is the dual frame, the vector-valued 1-form K of the proposition above may be written as $\sum_a \theta^a \otimes X_a$. Define $\Theta^a{}_b(X)$ as usual: $\Theta(X)X_b = \sum_a \Theta^a{}_b(X)X_a$. Then

$$T^{\nabla'}(X, Y) = \Theta(X)Y - \Theta(Y)X = \sum_a \big((\Theta^a{}_b \wedge \theta^b)(X, Y)\big)X_a.$$

Let $T^a \in \Omega^2(M)$ be defined as $\sum_b \Theta^a{}_b \wedge \theta^b$. Then we may write $\mathrm{d} = \sum_a(\theta^a \nabla'_{X_a} + T^a i_{X_a})$, and therefore

$$\langle H, \mathrm{d}\rangle = -\sum_{a,b,c} \Theta^a{}_b \wedge \theta^b \wedge \theta^c \omega_{ac} = -\sum_{b<c} (\Theta^a{}_b \omega_{ac} - \Theta^a{}_c \omega_{ab}) \wedge \theta^b \wedge \theta^c.$$

Thus $\langle H, \mathrm{d}\rangle$ vanishes (i.e., with $f = 1$) if and only if $\Theta^a{}_b \in \mathfrak{sp}(\omega)$—the symplectic algebra of ω. Using the definition of Θ in terms of the data ω and K_1, together with the property $K(Y, X, Z) - K(Z, X, Y) = (\nabla_X \omega)(Z, Y)$, the obstruction boils down to $\nabla\omega = 0$. Thus

$$\Theta \in \mathfrak{sp}(\omega) \iff \nabla\omega = 0 \iff \mathrm{d}\omega = 0.$$

ACKNOWLEDGEMENT: One of us (O.A.S.V.) would like to acknowledge the fruitful conversations with O. Gil-Medrano and V. Miquel. Also, he would like to thank the kind hospitality received from the Departament de Geometria i Topologia de la Universitat de València, as well as the financial support received both from this university and from the Ministerio de Educación y Ciencia of Spain, while on sabbatical leave from CIMAT.

References

[1] M. Batchelor, *The structure of supermanifolds*, Transactions of the American Mathematical Society **253** (1979), 329–338.

[2] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.

[3] P. Green, *On holomorphic graded manifolds*, Proceedings of the American Mathematical Society **85** (1982), 587–590.

[4] A. Frölicher and A. Nijenhuis, *Theory of vector valued differential forms*, Indagationes Mathematicae **18** (1956), 338–359.

[5] B. Kostant, *Holonomy and the Lie algebra in infinitesimal motions of a Riemannian manifold*, Transactions of the American Mathematical Society **80** (1955), 528–542.

[6] B. Kostant, *Graded Manifolds, Graded Lie Theory and Prequantization*, Lecture Notes in Mathematics **570**, Proceedings of the Conference on Differential Geometry Methods in Mathematical Physics, Bonn, 1975 (K. Bleuler and A. Reetz, eds.), Springer-Verlag, Berlin and New York, 1977, pp. 177–306.

[7] J. L. Koszul, *Connections and splittings of supermanifolds*, Differential Geometry and Its Applications **4** (1994), 151–161.

[8] J. Monterde, *A characterization of graded symplectic structures*, Differential Geometry and Its Applications **2** (1992), 81–97.

[9] J. Monterde and O. A. Sánchez-Valenzuela, *The exterior derivative as a Killing vector field*, Israel Journal of Mathematics **93** (1996), 157–170.

[10] M. Rothstein, *Deformations of complex supermanifolds*, Proceedings of the American Mathematical Society **95** (1985), 251–260.

[11] M. Rothstein, *The structure of supersymplectic supermanifolds*, Lecture Notes in Physics **375**, Proceedings of the Conference on Differential Geometry Methods in Mathematical Physics, Rapallo, 1990 (C. Bartocci, U. Bruzzo and R. Cianci, eds.), Springer-Verlag, Berlin and New York, 1991, pp. 331–343.

[12] M. Rothstein, *Equivariant splittings of supermanifolds*, Journal of Geometry and Physics **12** (1993), 145–152.

[13] G. Tuynman, *Geometric quantization of the BRST charge*, Communications in Mathematical Physics **150** (1992), 237–265.

[14] I. Vaisman, *Remarkable operators and commutation formulas on locally conformal Kähler manifolds*, Compositio Mathematica **40** (1980), 287–299.